On Hyperkähler and Quaternionic Kähler Geometry

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Abstract

We look at the relations between hyperkähler and quaternionic Kähler geometry. We examine, in particular, the existence of a correspondence between the two geometries when the underlying manifolds admit certain additional symmetries. We describe two ways in which such a correspondence can be realized and, eventually, we try to analyze what the correspondence gives in some explicit examples. We then explore the general theory of twistor spaces and provide an explicit construction of the twistor space of $\mathbb{P}^2$.

Contents

Introduction

In this joint work, we analyze some remarkable constructions about hyperkähler and quaternionic Kähler manifolds and a series of relations existing between these two geometries.

Under certain circumstances, there is a remarkable connection between hyperkähler and quaternionic Kähler geometry, supplied by Swann's constructions [13]: if, on one hand, over any quaternionic Kähler manifold with positive scalar curvature one can build a hyperkähler manifold endowed with a permuting action of $\mathbb{H}^*$, on the other hand, one can obtain, through an appropriate quotient, a quaternionic Kähler manifold from any hyperkähler manifold which admits a free, isometric and permuting $Sp(1)$-action such that the field IX_I does not depend on the choice of the complex structure I, where X_I is the Killing field generated by the $U(1) \subset Sp(1)$ preserving I. This link is at the heart of a correspondence existing between the two geometries when the considered manifolds admit in addition specific types of circle actions: more precisely, the correspondence takes into account, on one side, a hyperkähler manifold with a circle action preserving just one complex structure and, on the other side, a quaternionic Kähler manifold of the same dimension with a quaternionic Kähler circle action. In the first part of the project, we will then analyze the way to pass back and forth between the two sides, following first a differential geometric approach, based on Haydys's construction in [10], and then giving the general idea of an alternative construction based on twistor theory and developed by Hitchin in [11]. Eventually, we will try to understand how the correspondence works in some specific examples.

In the second half of the project, we start off by describing the construction of the twistor spaces for oriented Riemannian 4-manifolds, then we move on to describe the higher dimensional analogue for the hyperkähler and quaternionic Kähler case. Twistor theory allows us to encode information about the metric purely in terms of holomorphic data. The strength of this approach is that under certain circumstances the process is reversible i.e. we can construct the metric from just holomorphic data. We shall give an explicit description of the twistor space of the complex projective space, $\mathbb{P}^2$ and along the way give an overview of hermitian symmetric spaces.

More systematically, the structure of the paper is as follows.
The first section is devoted to summarize some fundamental constructions from hyperkähler and quaternionic Kähler geometry that will be needed in the second section. We briefly describe hyperkähler reductions and the Gibbons-Hawking Ansatz in hyperkähler geometry, and then we present the essential bundle and quotient constructions for quaternionic Kähler manifolds. Clarifying examples are provided. The second section is entirely devoted to the study of the link between the two geometries and in particular to the ways to construct a correspondence between hyperkähler and quaternionic Kähler manifolds endowed with the circle actions mentioned above. The attention will be concentrated in particular on the transition from the hyperkähler to the quaternionic Kähler side of the correspondence. In the third section, we try to give some examples of how the correspondence works in practice. On the hyperkähler side, we look at the case of $T^*\mathbb{CP}^n$, of the flat manifolds and of the Taub-NUT metric obtained as Gibbons-Hawking space; we also describe as the case of the Eguchi-Hanson metric is solved by Hitchin through the twistor theory. On the quaternionic Kähler side, one can try to look at the case of $\mathbb{CP}^2$. In the fourth section we first explain the general theory in the construction of twistor spaces pointing out the main differences in the four dimensional, hyperkähler and quaternionic Kähler cases. We give a quick description of the twistor space of the 4-sphere, S^4 and proceed to analyse the twistor space of the complex projective space, $\mathbb{P}^2$ which is the flag manifold, $\mathbb{F}^{(1,2,3)}$ in more details. We give an explicit construction of this twistor space and the associated structures by means of Cartan's moving frame technique by taking advantage of the fact that these are in fact symmetric spaces. We also find the nearly Kähler structure on $\mathbb{F}^{(1,2,3)}$.

1 Background on hyperkähler and quaternionic Kähler manifolds

We start giving a brief summary of some background material about hyperkähler and quaternionic Kähler geometry, which lays the foundation for the following arguments. The notations of this section will be then used in the rest of the presentation. We refer freely to well-known notions from Riemannian geometry, complex geometry and symplectic geometry.

1.1 Hyperkähler manifolds

Definition 1.1. *A hyperkähler manifold is a Riemannian manifold (M, g) with three covariantly constant almost complex structures $\{I, J, K\}$ compatible with the metric and satisfying the quaternionic identities: $IJ = K = -JI$.*

Another equivalent definition of hyperkähler manifold is the requirement to have holonomy group contained in $Sp(n)$.

Note that any manifold M enjoying a pair of anti-commuting almost complex structures has actually a whole collection $\mathcal{A}$ of almost complex structures parametrized by the sphere $S^2 \subset \mathrm{Im}\mathbb{H}$. For every $A \in \mathcal{A}$, one can define a non-degenerate 2-form $\omega_A(\cdot, \cdot) = g(A\cdot, \cdot)$ (or, $\omega_A(\cdot, \cdot) = g(\cdot, A\cdot)$ depending on the conventions), which can be shown to be closed. In fact, by defining a two form ω with values in the imaginary quaternions

$$\omega = \omega_I i + \omega_J j + \omega_K k,$$

the condition $\nabla I = \nabla J = 0$ is indeed equivalent to $d\omega = 0$. The integrability constraint is not part of the definition of hyperkähler manifold since it is a direct consequence of $d\omega = 0$.

The basic example of hyperkähler manifold is the flat space $\mathbb{H}^n$. The first-known non-flat example was instead the Eguchi-Hanson metric on $T^*\mathbb{CP}^1$. Calabi provided first example in dimension bigger than 4, by constructing a hyperkähler structure on $T^*\mathbb{CP}^n$ which coincides with the Eguchi-Hanson metric for $n = 1$. The first compact example was given by Yau's solution of the Calabi Conjecture. One can define hyperkähler structures also on moduli spaces of solutions of the Yang-Mills(-Higgs) equation (see [14]).

1.1.1 Hyperkähler reductions

One of the main routes to constructing hyperkähler metrics is via hyperkähler quotient. This method first appeared in a famous paper by Hitchin et al [6].

A vector field X on a hyperkähler manifold M with complex structures $\{I, J, K\}$ is called *tri-holomorphic* if $\mathcal{L}_X I = \mathcal{L}_X J = \mathcal{L}_X K = 0$ and *tri-hamiltonian* if $\mathcal{L}_X \omega = 0$. Note that, for any Killing field on M, these two conditions are equivalent.

Now, if G is a rank m compact Lie group of isometries acting freely on M and preserving ω (we say *triholomorphically*) and, further, the first cohomology group $H^1(M, \mathbb{R})$ vanishes, then we can define three moment maps μ_1, μ_2, μ_3 associated to the action, which can be condensed in a unique map

$$\mu : M \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$$

defined by $\langle \mu(m), \xi \rangle = \mu_{X_\xi}(m)$, where $\xi \in \mathfrak{g}$, X_ξ is the vector field generated by ξ on M and $\mu_{X_\xi}(m)$ is such that $d\mu_{X_\xi}(m) = X_\xi \lrcorner \omega$. Note that one can always choose μ to be equivariant (since we have required G to be compact).

Take now a point $c \in \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ invariant under the coadjoint action of G. Then the level set

$M_c = \mu^{-1}(c)$ is a submanifold of M of dimension $(4n-3m)$, which is also G-invariant and on which G acts freely, isometrically and preserving ω. Hitchin et al. showed that, under the requirements above, the hyperkähler metric on M descends to a hyperkähler metric on the $(4n-4m)$-dimensional manifold $\tilde{M}_c = M_c/G$.

An example
Take $M = \mathbb{H}^n$ and take as triple of 2-forms the anti-self dual basis. Then $\omega = \frac{1}{2} d\bar{q}^t \wedge dq$, where $q : \mathbb{H}^n \to \mathbb{H}^n$ is the identity. Let now $S^1 \subset Sp(n)$ act diagonally on $\mathbb{H}^n$ via

$$\mathrm{e}^{\mathrm{i}\theta} \cdot \mathrm{q} = \begin{pmatrix} e^{i\theta} \cdot q_1 \\ \cdots \\ \cdots \\ e^{i\theta} \cdot q_n \end{pmatrix}$$

where $e^{i\theta} \cdot q_k = e^{i\theta}(q_{k0} + q_{k1}i) + e^{i\theta}(q_{k2} + q_{k3}i)j = e^{i\theta}(q_{k0} + iq_{k1}) + je^{-i\theta}(q_{k2} - iq_{k3}i)$. Then one has $X_i(q) = iq$ and

$$X_i \lrcorner \Omega = \frac{1}{2} d\bar{q}^t \wedge d\bar{q}\ (iq) = -\frac{1}{2}\bar{q}^t i dq - \frac{1}{2} d\bar{q}^t iq = \frac{1}{2} d(-\bar{q}^t iq).$$

Hence, the moment map can be identified with $\mu(q) = \bar{q}^t iq$ and we obtain a hyperkähler quotient for every $c \in \mathrm{Im}\mathbb{H}$. For $c \neq 0$, all these quotients are each other homothetic and, in particular, homothetic to the one at level i : indeed, given $c \neq 0$, one can always find $h_c \in \mathbb{H}$ such that $\bar{h}_c i h_c = c$ and the map $\phi : \mathbb{H}^n \to \mathbb{H}^n$ sending $q \mapsto qh_c^{-1}$, which is $U(1)$-equivariant, brings the set $\{\mu = c\}$ into the set $\{\mu = i\}$

$$\mu(\phi(q)) = \mu(qh_c^{-1}) = \overline{qh_c^{-1}}^t\ i\ qh_c^{-1} = \overline{h_c^{-1}} c h_c^{-1} = i,$$

so that the following diagram commutes

$$\begin{tikzcd} \mu^{-1}(c) \arrow[r, "\phi"] \arrow[d, "\pi"] & \mu^{-1}(1) \arrow[d, "\pi"] \\ M_c \arrow[r, "\hat{\phi}"] & M_i \end{tikzcd}$$

Now, writing q as $z_1 + jz_2$ as above and expanding $i = \bar{q}^t iq$, one finds the relations

$$\begin{cases} |z_1|^2 - |z_2|^2 = 1 \\ z_2^t z_1 = 0. \end{cases}$$

We can think at z_1 as defining a point in $\mathbb{CP}^{n-1}$ and at $z_2 \otimes \bar{z}_1$ as an element in $(\mathrm{Hom}(\langle z_1 \rangle, \langle z_1 \rangle^\perp))^*$: in fact, from the relation $z_2^t z_1 = 0$, one has that $z_2 \in \langle \bar{z}_1 \rangle^\perp$, thus, taking $\varphi \in \mathrm{Hom}(\langle z_1 \rangle, \langle z_1 \rangle^\perp)$, one can define naturally

$$z_2 \otimes \bar{z}_1(\varphi) = z_2^t \varphi(z_1);$$

this pairing is not degenerate since, if we had $z_2^t \varphi(z_1) = 0\ \forall \varphi$, then $z_2^t w = 0\ \forall w \in \langle z_1 \rangle^\perp$, but also $z_2^t z_1 = 0$, so that $z_2^t w = 0\ \forall w$; choosing $w = \bar{z}_2$, one has $z_2 = 0$. Since we know that $(\mathrm{Hom}(\langle z_1 \rangle, \langle z_1 \rangle^\perp))^* \cong T^*_{[z_1]}\mathbb{CP}^{n-1}$, we can conclude that, topologically, the quotient at every $c \neq 0$ is $T^*\mathbb{CP}^{n-1}$. The induced hyperkähler metric agrees with the Calabi one.

1.1.2 Gibbons-Hawking Ansatz

The Gibbons-Hawking Ansatz gives a system of coordinates which is particularly appropriate to describe a hyperkähler metric with a triholomorphic S^1 action. We follow here the construction in [12]. An alternative reference can be [15].

Consider an open set $U \subset \mathbb{R}^3$ with the Euclidean metric and with coordinates u_1, u_2, u_3 and let $\pi : X \to U$ be a S^1-principal bundle, with S^1-action generated by the vector field $\frac{\partial}{\partial t}$. Let θ be a connection 1-form on X, (namely, a $i\mathbb{R}$-valued S^1-invariant 1-form on X such that $\theta(\frac{\partial}{\partial t}) = i$. Its curvature is $d\theta = \pi^*\alpha$ for a $2-$form α on U, and the first Chern class of $\pi : X \to U$ is given $\frac{1}{2\pi i}\alpha$. Now, assume that V is a real function on U satisfying

$$\star dV = \frac{\alpha}{2\pi i}. \tag{1}$$

(Note that V is harmonic since $d\alpha = 0$ implies $\star d \star dV = 0$.) Set $\theta_0 = \frac{\theta}{2\pi i}$ and

$$\begin{cases} \omega_1 = du_1 \wedge \theta_0 + V \; du_2 \wedge du_3 \\ \omega_2 = du_2 \wedge \theta_0 + V \; du_3 \wedge du_1 \\ \omega_3 = du_3 \wedge \theta_0 + V \; du_1 \wedge du_2. \end{cases} \tag{2}$$

If we assume that V is positive everywhere on U, then $\omega_1, \omega_2, \omega_3$ are non-degenerate, $\omega_i \wedge \omega_j = 0$ if $i \neq j$ and (1) implies that $d\omega_i = 0$ for every i. Then one can directly verify that $\omega_1,\ \omega_2,\ \omega_3$ define a hyperkähler structure on X (and π turns out to be the hyperkähler moment map for the S^1 action). Indeed, taking for instance $\Omega = \omega_2 + i\omega_3$ to be the (holomorphic) 2-form on X determines an integrable complex structure I on X, with $du_2 + du_3$ and $\theta_0 - iV du_3$ spanning the holomorphic cotangent space. Now, putting $g(v, w) = \omega(v, Iw)$, one finds for the metric

$$g = V(du_1^2 + du_2^2 + du_3^2) + \frac{1}{V}\theta_0^2. \tag{3}$$

In fact, the standard procedure is to take a positive harmonic V such that $\star dV$ gives the Chern class of the bundle. Then θ is uniquely determined by the condition 1 (up to pull-back closed 1-forms from the open set U).

Taub-NUT metric and instantons Consider the Hopf fibration $\pi : S^3 \subset \mathbb{C}^2 \to S^2 \subset \mathbb{R}^3$, extend it to a map $\mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$, and compose it with the complex coniugation on z_2 to get a map $\mu : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$

$$(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2\mathrm{Re}(z_1 z_2), 2\mathrm{Im}(z_1 z_2)). \tag{4}$$

The map describes X as S^1-bundle over $\mathbb{R}^3 \setminus \{0\}$, with Chern class ± 1 and S^1 action given by $e^{it} \cdot (z_1, z_2) = (e^{it} z_1, e^{-it} z_2)$.
Notice that, via the identification $q = (q_0 + iq_1) + j(q_2 - iq_3) = (z_1, z_2)$, this is exactly the moment map $\mu : \mathbb{H} \to \mathrm{Im}\mathbb{H} \cong \mathbb{R} \oplus \mathbb{C} : q \mapsto \bar{q}iq = u_1 i + k(u_2 + iu_3)$ for the action of S^1 on $\mathbb{H}$ considered above: $(e^{it}, q) \mapsto e^{it} \cdot q$. We can choose

$$\theta = i \, \frac{\mathrm{Im}(\bar{z}_1 dz_1 - \bar{z}_2 dz_2)}{|z_1|^2 + |z_2|^2}. \tag{5}$$

Then a positive harmonic function on $\mathbb{R}^3 \setminus \{0\}$ satifying (1) has to be of the form

$$V_e = e + \frac{1}{4\pi|u|} \quad (e \geq 0, u = (u_1, u_2, u_3)). \tag{6}$$

We have found in this way a family of hyperkähler structure on the space X, given by $g_e = V_e \, du \cdot du + \frac{1}{V_e}\theta_0^2$, which, for $e \geq 0$, extend to all $\mathbb{R}^4$. When $e = 0$, g_e is just the flat metric on $\mathbb{C}^2$, when the choice $e > 0$ gives so-called Taub-NUT metric, whose asymptotic behaviour at infinity is completely different from the Euclidean case: the volume of large geodesic balls in Taub-NUT goes asymptotically as r^3. Such metrics are in fact ALF (asymtotically locally flat) spaces, which approach a flat metric for $|u|$ going to ∞, but being periodic in the coordinate t.
Further, choosing in the Ansatz

$$V_e = e + \sum_{i=1}^{n} \frac{1}{4\pi|u - u_i|}, \tag{7}$$

where $u_1, \ldots u_n \in \mathbb{R}^3$ are n distinct points in $\mathbb{R}^3$, gives an infinite family of ALF spaces. With the choice $e = 0$, we find as before complete hyperkähler structures of type ALE (asymptotically locally Euclidean), of volume growth $r^4 : k = 1$ gives $\mathbb{C}^2$ and the flat metric, $k = 2$ is Eguchi-Hanson and the following ones go under the name of multi Eguchi-Hanson metrics. For a treatment of this topic, see [14].

1.2 Quaternionic-Kähler manifolds

In the next exposition, we follow quite closely Swann's presentation of quaternionic-Kähler manifolds in [13]. We first start by recalling some fundamental definitions.

Definition 1.2. *A $4n$-Riemannian manifold (N, g) is called* quaternionic-hermitian *if there exists a sub-bundle $\mathcal{A} \subset \mathrm{End}(TN)$ spanned by a basis I, J, K of almost complex structures satisfying the quaternionic identit $K = IJ = -JI$ in a neighbourhood U_x of every point $x \in N$ and the metric g is compatible with $\mathcal{A}$, namely $g_x(A\cdot, A\cdot) = g_x(\cdot, \cdot)$ $\forall A \in \mathcal{A}_x$ and $x \in N$.*

We want to remark that, in this case, unlike the hyperkähler case, we have a basis of almost complex structures only locally.

The bundle $\mathcal{A}$ can be naturally embedded in $\Lambda^2 T^*N$, through the identification $A_x \mapsto (\omega_A)_x$. Further, although a basis $\{I, J, K\}$ of $\mathcal{A}$ can be defined just locally, the so-called *fundamental 4-form* of N, locally defined via

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K,$$

is a non-degenerate 4-form which turns out to be globally well defined.

Definition 1.3. *For $n > 1$, a $4n$-dimensional quaternionic-hermitian manifold is called* quaternionic Kähler *if $\nabla\Omega = 0$, where ∇ denotes the Levi-Civita connection of g.*

If $\dim N \geq 12$, then $\nabla\Omega = 0$ is equivalent to $d\Omega = 0$. If $\dim N = 8$, N is quaternionic Kähler if $d\Omega = 0$ and, in addition, the algebraic ideal generated by $\mathcal{A}$ is differential (i.e., closed under differential d). A proof of these results can be found in [13, Chapter 5].

An equivalent definition of quaternionic Kähler manifold is the request for the holonomy group of the N to be contained in the subgroup

$$Sp(n)Sp(1) = Sp(n) \times_{\mathbb{Z}_2} Sp(1)$$

of $SO(4n)$, which implies (Aleskseevskii, 1968) that M is an Einstein manifold.

If N is 4-dimensional, the condition $\nabla\Omega = 0$ is automatically satisfied, since Ω is nothing but six times the volume form of N, so an extension of the previous definition is needed. In that case, a Riemannian manifold is said to be *quaternionic Kähler* if it is oriented, Einstein and self-dual.

In what follows, we will consider exclusively manifolds with non-zero scalar curvature, since simply connected quaternionic Kähler manifolds with zero scalar curvature are easily seen to be hyperkähler manifolds. Our model example of a quaternionic Kähler manifold is the quaternionic projective space $\mathbb{H}P^n$, with its symmetric metric.

1.2.1 The associated Swann bundle

If N is any quaternionic Kähler $4n$-manifold, one can construct naturally a $(4n+4)$-manifold associated to it. Take the reduced frame bundle F of N, i.e., for every $x \in N$, consider the collection of frames $u : \mathbb{H}^n \to T_xN$ compatible with the structure of quaternionic manifold. Locally, F can be lifted to a principal $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$- bundle $\tilde{F}$ which double covers F, allowing the construction of bundles associated to representations of $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$. A global lift of F to $\tilde{F}$ is not always possible, but can be realized just when the cohomology class $\epsilon \in H^2(M, \mathbb{Z}_2)$ defined by Romani and Marchiafava vanishes.
Consider now the bundle H obtained by quotienting $\tilde{F} \times \mathbb{H}$ by $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$:

$$\begin{tikzcd} \tilde{F} \times \mathbb{H} \arrow[r, "\varpi"] \arrow[d, "\pi_1"] & H \arrow[d, "\pi_H"] \\ \tilde{F} \arrow[r, "\pi"] & N \end{tikzcd}$$

Starting from the structure on N, we will briefly see how one may define hyperkähler and quaternionic Kähler metrics on H.
Consider the canonical 1-form $\theta \in \Omega^1(F, \mathbb{H}^n)$ given by $\theta_u(v) = u^{-1}(d\pi(v))$, where $u \in F$ and $v \in T_uF$, and the torsion-free connection $\omega \in \Omega^1(F, sp(n) \oplus sp(1))$ induced by the Levi-Civita connection on N, which can be decomposed as $\omega = \omega_+ + \omega_-$, relative to the decomposition

of the Lie algebra. Now pull-back to $\tilde{F} \times \mathbb{H}$ the forms $\theta, \omega_\pm$ and the identity $x : \mathbb{H} \to \mathbb{H}$ and define

$$\alpha = dx - x\omega_- \ \in \Omega^1(\tilde{F} \times \mathbb{H}, \mathbb{H}).$$

One can prove that the forms $x\bar{\theta}^t \wedge \theta\bar{x}$ and $\alpha \wedge \bar{\alpha}$ on $\tilde{F} \times \mathbb{H}$ are indeed pull-backs of forms on H, and also the function $r^2 = x\bar{x}$ is well defined on H. Thus we may consider the 2-form $\nu \in \Omega^2(H, \mathrm{Im}\mathbb{H})$ given by

$$\nu = f(r^2)\alpha \wedge \bar{\alpha} + g(r^2)x\bar{\theta}^t \wedge \theta\bar{x},$$

whose i, j, k components are non-degenerate if f, g are nowhere-zero real-valued functions. Further, the pseudo-Riemannian metric g on H

$$g = Re(f(r^2)\alpha \otimes \bar{\alpha} + g(r^2)r^2\bar{\theta}^t \otimes \theta),$$

turns out to be positive definite away from the zero section if $f, g > 0$ everywhere and, in addition, the almost complex structures associated to g and ν do not depend on f and g.
If we put now $\Upsilon = \nu \wedge \nu$, then g is pseudo-hyperkähler if $d\nu = 0$, pseudo-quaternionic Kähler if $d\Upsilon = 0$. This translates into appropriate conditions on f, g, as shown by Swann in [13, Theorem 2.1.5]. We obtained in this way pseudo-hyperkähler and pseudo-quaternionic Kähler metrics on $H \setminus 0$.
However, as mentioned above, in the general case H does not exist globally over N; therefore, in the generic situation, one needs to consider instead the bundle

$$\mathcal{U}(N) = F \times_{Sp(n)Sp(1)} (\mathbb{H}^*/\mathbb{Z}_2).$$

This bundle, the *Swann bundle*, is a principal $\mathbb{H}^*/\mathbb{Z}_2$ bundle over N, with action induced from the action $(q, h) \mapsto \bar{q}h$ of $\mathbb{H}^*$ on $\mathbb{H}$. The results seen before still hold on the Swann bundle, as the form and the metric considered above are invariant through the quotient and thus well-defined when projected to $\mathcal{U}(N)$. In particular, the metrics defined on $\mathcal{U}(N)$ are positive definite if N has positive scalar curvature.

An example
Consider the projective space $\mathbb{HP}^n$, the Swann bundle $\mathcal{U}(\mathbb{HP}^n)$ can be directly derived. Indeed, using the fact that $\mathbb{HP}^n \cong Sp(n+1)/(Sp(n) \times Sp(1))$, one can shows that, in this case, the lift $\tilde{F}$ is exactly $Sp(n+1)$ and so

$$H \setminus 0 = \tilde{F} \times_{Sp(n)\times Sp(1)} \mathbb{H}^* = \frac{Sp(n+1)}{Sp(n)} \times_{Sp(1)} \mathbb{H}^* = \frac{Sp(n+1)}{Sp(n)} \times \mathbb{R}_{>0} = S^{4n+3} \times \mathbb{R}_{>0},$$

thus $H \setminus 0$ is $\mathbb{H}^{n+1^*}$ topologically. Further, the hyperkähler metric on it can be shown to be the flat metric (see [13]), which may be completed by adjoining a point.
(The quaternionic Kähler metrics, instead, are induced by the inclusion $\mathbb{H}^{n+1^*} \subset \mathbb{H}P^{n+1}$ and are completed by adjoining a copy of $\mathbb{H}P^n$ at infinity.)

1.2.2 Quotients

The quotient constructions for symplectic, Kähler and hyperkähler manifolds have been extended to the quaternionic Kähler case by Galicki and Lawson.

Quaternionic Kähler actions on N A Killing vector field W on a quaternionic Kähler manifold N is said to be *quaternionic* if $\mathcal{L}_W\Omega = 0$. One can prove that (see [13, Lemma 3.1.1])

Lemma 1.4. *If W is a quaternionic vector field on N, then it can be lifted to a Hamiltonian Killing field $\tilde{X}$ on $\mathcal{U}(N)$.*

It follows then that, if a group G acts on N freely and isometrically and preserving Ω (we say *quaternionically*), its action can be lifted to an isometric and triholomorphic action on $\mathcal{U}(N)$, that is also free, since the fibres of $\pi_H : \mathcal{U}(N) \to N$ are preserved.

Consider now a Killing field Y generated by G, its lift X to $\mathcal{U}(N)$ and the $Sp(n)Sp(1)$-invariant vector field on F, still indicated by X, satisfying $\varpi(X) = X$. Then the function μ_X on $F \times \mathbb{H}^*$ given by the espression

$$\mu_X = -X \lrcorner (x\omega_- \bar{x})$$

is $Sp(n) \times Sp(1) \times \mathbb{Z}_2$-invariant and well-defined on $\mathcal{U}(N)$ and one can prove [13, Proposition 3.1.2]

Proposition 1.5. *The map $\mu : \mathcal{U}(N) \to \mathfrak{g}^* \otimes \mathrm{Im}\mathbb{H}$ defined above is a hyperkähler moment map for the induced action of G on $\mathcal{U}(N)$.*

Consider now $\mathcal{A}$ embedded in $\Lambda^2 T^*N$, which is nothing but the subbundle of $\Lambda^2 T^*N$ (denoted again with $\mathcal{A}$) with fiber $\mathfrak{sp}(1) \cong \mathrm{Im}\mathbb{H}$. Taken the 1-form α_Y obtained by contracting Y with the metric g, consider $\nabla\alpha_Y \in \Lambda^2 T^*N$ and its the orthogonal projection $(\nabla\alpha_Y)^{\mathcal{A}}$ onto $\mathcal{A}$. Choosing a frame $u \in F$, we have that

$$(\nabla\alpha_Y)^{\mathcal{A}}_{\pi(u)} = \omega_-(X)_u.$$

Thus, μ_X is zero at $c \in \mathcal{U}(N)$ if and only if $(\nabla\alpha_Y)^{\mathcal{A}}_{\pi_H(c)} = 0$, so that μ_X is zero on the entire fibre along c when vanishing at c.

Quaternionic Kähler reductions Let G be a compact Lie group acting freely, smoothly, isometrically and quaternionically on N. For any quaternionic Killing field Y on N generated by the action, let

$$f_Y = \frac{1}{\lambda}(\nabla\alpha_Y)^{\mathcal{A}},$$

where λ is the Einstein constant for N.

Definition 1.6. *(Galicki, 1987) The moment map $N \to \mathfrak{g}^* \otimes \mathcal{A}$ associated to the action is defined by*

$$\langle \phi(m), Y \rangle = f_Y(m),$$

for every $m \in M$ and every vector field Y generated by G.

One can prove (see [13, Lemma 3.2.2]) that ϕ is G-equivariant; in this case, however, $N_0 = \phi^{-1}(0)$, where 0 is the zero section, is the only one natural G-invariant submanifold we can consider. Galicki and Lawson (1988) proved that $N_G = N_0/G$ has a quaternionic Kähler structure derived from N.

An example
The action of $U(1)$ on $\mathbb{H}^{n+1}$ previously considered descends to $\mathbb{H}P(n)$ and the moment map at $[q_0 : \cdots : q_n]$ is still given by $\bar{q}iq$. Writing $q = a + jb$, there is $A \in \mathrm{U}(n+1)$ such that $Aa = (1, 0, \dots, 0)^t$; so, if $\phi(q) = 0$, then also $Aq = Aa + j\bar{A}b$ is a zero and $B^t \bar{A}^t Ab = b^t a = 0$, thus $\bar{A}b = (0, b_1, \dots, b_n)^t$ and we can choose it to be $(0, 1, \dots, 0)^t$. Hence, $U(n+1)$ acts transitively on the zero set of ϕ; the quaternionic Kähler quotient is then

$$\frac{U(n+1)}{SU(2) \times U(n-1) \times U(1)} = \mathrm{Gr}_2(\mathbb{C}^{n+1}),$$

where $SU(2) \times U(n-1)$ is the stabiliser at one point.

Commuting quotients If one takes the hyperkähler reduction of the Swann bundle $\mathcal{U}(N)$ at the zero level set of the moment map, the following remarkable property of commutativity holds (see [13, Theorem 3.3.1]):

Theorem 1.7. *If N is a quaternionic Kahler manifold and G acts isometrically, freely and quaternionically on N, then the pseudo-hyperkähler quotient of $\mathcal{U}(N)$ by the lifted G action is exactly the Swann bundle associated to the quaternionic Kähler quotient of N by G.*

2 The correspondence

In this section we analyze the link between hyperkähler and quaternionic Kähler geometry and, in particular, the existence of a correspondence between hyperkähler and quaternionic Kähler manifolds admitting particular types of symmetries. More precisely, we will consider, on one side, a hyperkähler manifold with a circle action preserving just one complex structure and, on the other side, a quaternionic Kähler manifold of the same dimension with a quaternionic Kähler circle action. We will describe how one could go from the first side to the second one and viceversa and we will try to undestrand to which extent the construction is symmetrycal.

The transition from the quaternionic Kähler to the hyperkähler side requires the construction of the Swann bundle of the quaternionic manifold, while the opposite one involves the introduction of a natural S^1 bundle over the hyperkähler manifold whose connection turns out to be hyperholomorphic; those contructions have been classically developed through two different approaches: the first of them is of differential geometric type and is essentially based on Haydys's results in [10], while the second, by Hitchin [11], makes use of twistor theory.

Here we will mainly follow the differential geometric perspective, taking some ideas from Hitchin's construction when explaining the transition from the hyperkähler to the quaternionic Kähler side. Explicit examples of the construction will be presented in the next chapter.

2.1 The hyperkähler/quaternionic Kähler link

There is a noteworthy link between hyperkähler and quaternionic Kähler geometry, provided by Andrew Swann's constructions.

Permuting complex structures Let M be a hyperkähler manifold with the relative 2-form $\omega \in \Omega^2(M, \mathrm{Im}\mathbb{H})$ and assume that $\mathrm{SU}(2) = \mathrm{Sp}(1)$ acts on M isometrically via $(q, m) \mapsto \phi(q, m)$. We say that the action of $\mathrm{Sp}(1)$ *permutes complex structures* (or is *permuting*) if

$$\phi_q^*(\omega) = q\omega\bar{q} \quad \forall\, q \in \ \mathrm{Sp}(1). \tag{8}$$

which implies that $\mathcal{L}_{X_\xi}\omega = [\xi, \omega] = \xi \cdot \omega - \omega \cdot \xi$, with $\xi \in sp(1) = \mathrm{Im}\mathbb{H}$ and X_ξ the associated vector field on M. In fact,

$$\begin{aligned} \mathcal{L}_{X_\xi}\omega = \frac{\partial}{\partial t}_{|t=0}(\phi_t^\xi)^*\omega = \lim_{t\to 0}\frac{(\phi_t^\xi)^*\omega - \omega}{t} \\ = \lim_{t\to 0}\frac{\exp t\xi \cdot \omega \cdot \overline{\exp t\xi} - \omega}{t} = \frac{\partial}{\partial t}_{|t=0}\exp t\xi \cdot \omega \cdot \overline{\exp t\xi} \\ = \xi \cdot \omega + \omega \cdot \bar{\xi} = \xi \cdot \omega - \omega \cdot \xi, \end{aligned}$$

where $\exp t\xi$ is the integral curve of ξ in $\mathrm{Sp}(1)$ starting at 1 and ϕ^ξ is the flow generated by ξ on M.

It is clear that in this case the quotient constrcutions seen above do not apply. However, if we fix one complex structure, say I, then there is a subbundle $U(1) \subset Sp(1)$, (the stabilizer $Stab_i$ of i via $(q, h) \to qh\bar{q}$) which preserves I and rotates J and K : one has

$$\exp{(it)}\omega\exp{(-it)} = \omega_I i + \exp{(2it)}[(\omega_J + \omega_K i)j]$$

and $\mathcal{L}_{X_i}\omega = i\omega - \omega i = -2\omega_K j + 2\omega_J k$, so

$$\mathcal{L}_{X_i}\omega_I = 0, \quad \mathcal{L}_{X_i}\omega_J = -2\omega_K, \quad \mathcal{L}_{X_i}\omega_K = 2\omega_J.$$

Then the following result [13, Theorem 3.5.1] holds:

Theorem 2.1. *Let M be hyperkähler manifold which admits an isometric $Sp(1)$-action such that*

1) *there is a finite subgroup Γ of $Sp(1)$ such that $Sp(1)/\Gamma$ acts freely;*

2) *$Sp(1)$ induces a permuting action on the 2-sphere of complex structures;*

3) *if X_I is the Killing field generated by the circle subgroup of $Sp(1)$ preserving the structure I, then the real span of IX_I in TM is independent of the choice of I.*

Choose a subgroup $U(1) \subset Sp(1)$ preserving a complex structure I and let $\mu : M \to \mathbb{R}$ be a moment map for this action of the subgroup with respect to the Kähler structure defined by I. Then $\mu^{-1}(x)$ is $Sp(1)$- invariant and $\mu^{-1}(x)/Sp(1)$ is a quaternionic Kähler manifold.

Hyperkähler potentials In [6] Hitchin et al. proved that, if M is a hyperkähler manifold with a S^1 action preserving I and permuting J and K, then the moment map μ_I associated to the action of I is a Kähler potential the other two complex structures. One can then wonder under which conditions there exists a hyperkähler potential on M, i.e. a Kähler potential for all the complex structures simultaneously. In fact, one has that ([13, Proposition 3.6.2]

Theorem 2.2. *If a hyperkähler manifold M has a hyperkähler potential, then then M admits a local $Sp(1)$-action permuting complex structures and such that the vector field IX_I independent of I.*
Conversely, if M admits such an action, then M has a hyperkähler potential given by the moment map μ_I associated to I.

Note that, if M is a hyperkähler manifold with hyperkähler potential μ, then the vector field W dual to $d\mu$ is an infinitesimal quaternionic and it is exactly IX_I. Hence, the local $Sp(1)$-action above can be actually extended to a local homothetic action of the entire $\mathbb{H}^* = \mathbb{R}^* \times Sp(1)$ (also called *permuting*). In particular,

$$IY_I = JY_J = KY_K = -Y_0, \tag{9}$$

where Y_0 generates the homotetic action of $\mathbb{R}^* \subset \mathbb{H}^* : (\phi_r)^* g = r^2 g$.

We have then seen that, on one hand, for any quaternionic Kähler manifold N, one can construct a hyperkähler manifold $\mathcal{U}(N)$ with an induced action of $\mathbb{H}^*$; this action turns out to permute the complex structures over $\mathcal{U}(M)$. On the other hand, given any hyperkähler manifold M as in 2.1, the quotient $N = M/\mathbb{H}^*$ carries a quaternionic Kähler structure.

The hyperkähler 4-manifolds admitting a permuting S^1-action were classified by Gibbons, Pope, Atiyah and Hitchin and consist of the flat metric on $\mathbb{H}$, the TAUB NUT metric and the hyperkähler metric on the moduli space of charge 2-monopoles (see[14]), but only the flat one has a hyperkähler potential.

We are now ready to look at the influence of additional S^1-symmetries on the relations between hyperkähler and quaternionic Kähler manifolds.

2.2 S^1 symmetries: the quaternionic Kähler side

Let us start by looking at the quaternionic Kähler side.
Let N be a quaternionic Kähler manifold of positive scalar curvature and dimension $4n$ endowed with a quaternionic Kähler action of S^1. As we have seen in the previous chapter, one can construct over N a manifold $\mathcal{U}(N)$ of dimension $4n + 4$, the Swann bundle, which can be endowed with a hyperkähler structure and has a homotetic action of $\mathbb{H}^*$. Then, the S^1 action can be lifted to $\mathcal{U}(N)$ and the lifted action enjoys the properties of being isometric,

triholomorphic and free, as seen before. By taking the hyperkähler reduction of $\mathcal{U}(N)$ by this action, one obtains a hyperkähler manifold $\mathcal{K}(N)$ of dimension $4n$.

$$\begin{tikzcd}
 & \mathbb{H}^* \arrow[r] & \mathcal{U}(N) \arrow[dr] & S^1 \arrow[l] & \\
S^1 \arrow[r] & N \arrow[ur] & & \mathcal{K}(N) & Stab_i \arrow[l]
\end{tikzcd}$$

2.3 S^1 symmetries: the hyperkähler side

Let $\tilde{M}$ be a hyperkähler manifold endowed with an isometric and permuting complex structures action of S^1(denoted by S^1_r from now on). As mentioned above, to pass to the quaternionic side, one needs the introduction of a principal S^1-bundle P over $\tilde{M}$ (we denote this S^1 by S^1_0 from now on), to which the S^1_r action on $\tilde{M}$ must be extended.

We first present, following Haydys's construction in [10], the differential geometric way to obtain the bundle P (via its curvature form), which involves also the construction, via P itself, of another bundle over $\tilde{M}$, with fiber $\mathbb{H}^*$.

2.3.1 The differential geometric approach

Since there is a standard method of producing the desired bundle P when the hyperkähler manifold considered is espressed as the hyperkähler quotient of a manifold M, this will be the first case we consider.
Further, the construction that follows permits in addition to reconstruct an open and everywhere dense submanifold of the initial hyperkähler manifold M as a bundle over the reduction $\tilde{M}$.

$\tilde{M}$ as hyperkähler quotient Consider a hyperkähler manifold M, with metric g and symplectic structure $\omega = \omega_I i + \omega_J j + \omega_K k$, where $\omega_A(\cdot, \cdot) = g(\cdot, A\cdot)\ \forall A \in \mathcal{A}$, endowed in addition with

1) a isometric $\mathbb{H}^*$ action which permutes complex structures;

2) an additional hyperkähler action of S^1 (denoted with S^1_0 from now on), with moment map $\mu : M \to \mathrm{Im}\mathbb{H}$ given by $d\mu = -K_0 \lrcorner \omega$, where K_0 is the Killing vector field of the action.

Suppose that these two actions are commuting and that μ is equivariant with respect to $\mathbb{H}^*$:

$$\mu \circ L_x = x\mu\bar{x}, \quad x \in \mathbb{H}.$$

If we now consider the imaginary quaternion i and the relative level set of the moment map $P = \mu^{-1}(i)$, then the map

$$f : \mathbb{H}^* \times P \to M \setminus \mu^{-1}(0)$$
$$(x, m) \to xm$$

is surjective, thanks to the $\mathbb{H}^*$-equivariance of μ and since the action of $\mathbb{H}^*$ on $\mathrm{Im}\mathbb{H} \setminus \{0\}$ via $(x, h) \to xh\bar{x}$ is transitive. Indeed, given $n \in M$ such that $\mu(n) = h \neq 0$, one can find $x \in \mathbb{H}^*$ such that $h = xi\bar{x}$, thus, by defining $m = x^{-1}n$, we have that $\mu(m) = x^{-1}\mu(n)\overline{x^{-1}} = i$, thus m lies in P, and clearly $f(x, m) = n$. We denote by M_0 the open and everywhere dense submanifold $M \setminus \mu^{-1}(0)$ of M, for brevity.
Nevertheless, it is clear that f is not a bijection, since, thanks to the $\mathbb{H}^*$ equivariance of μ, the action of $Stab_i = S^1 \subset \mathbb{H}^*$ (which we denote by by S^1_r to avoid confusion) passes to the level set P :

$$\mu(\exp(it)m) = \exp(it)\mu(m)\exp(it) = \exp(it)i\exp(-it) = i.$$

Consequently, two points of the form (x, m) and $(xz, \bar{z}m)$, with $z \in S^1_r$, are both mapped by f into the same point in M_0 : $f(x, m) = xm = f(xz, \bar{z}m)$. What one obtains is then that the manifold M_0 can be identified with the quotient $\mathbb{H}^* \times_{S^1_r} P$:

$$M_0 \cong \mathbb{H}^* \times_{S^1_r} P.$$

Note now that, thanks to the $\mathbb{H}^*$-equivariance of μ, each non-zero imaginary quaternion is a regular value of μ; hence, if we suppose in addition that the action of S^1_0 is free on P, the quotient $\tilde{M} = P/S^1_0$ is nothing but the hyperkähler reduction of M and P can be described as a S^1_0-principal bundle over $\tilde{M}$.

What Haydis showed in [10] is that, under these circumstances, it is actually possible to express the hyperkähler structure of M_0 solely in terms of $\mathbb{H}^*$ and P. Moreover, since the structure on P will just depend to the one on $\tilde{M}$, on the Killing field K_0 and on the induced connection on P itself, the hyperkähler structure of M_0 will just depend on the same quantities and $\mathbb{H}^*$. The construction is very explicit and goes as follows.

Denote the squared norm of K_0 by $v^{-1} = g(K_0, K_0) : \tilde{M} \to \mathbb{R}_{>0}$, (notice that v^{-1} is defined on $\tilde{M}$ since S^1_0 acts isometrically on M). The connection induced by the action of S^1_0 on P is then given by

$$\xi(\cdot) = vg(K_0\cdot, \cdot) \in \Omega^1(P),$$

so that, for every vector u in $T\tilde{M}$, it is now defined a horizontal lift $\hat{u}$. Further, since

$$T.P \cong \mathbb{R}K_0 \oplus T.\tilde{M},$$

the metric on P is completely determined by the data of ξ, v and the induced metric $\tilde{g}$ on $\tilde{M}$:

$$g = \tilde{g} + v^{-1}\xi^2. \tag{10}$$

It is now important to observe that, since we asked for the the two actions on M to be commuting, the inherited action of S^1_r on P descends to $\tilde{M}$, with the property (inherited from M), of fixing I and rotating the plane generated by J and K. We indicate with K_r the Killing vector field of S^1_r on $\tilde{M}$ and its squared norm with $w = \|K_r\|_{\tilde{g}} : \tilde{M} \to \mathbb{R}_{>0}$. Eventually, we denote by

$$\eta = K_r \lrcorner \tilde{g} + K_r \lrcorner \tilde{\omega} \in \Omega^1(\tilde{M}, \mathbb{H}).$$

We now want to write down the relation between K_r, Killing vector field for the S^1_r on $\tilde{M}$, and Y_I, Killing vector field for the same action on P. First of all, we start observing that

$$T.M = T.P \oplus \mathbb{R}IK_0 \oplus \mathbb{R}JK_0 \oplus \mathbb{R}KK_0 \qquad \text{and} \qquad T.P = \mathrm{Ker}d\mu.$$

Moreover, $d\mu(IK_0) = -\omega(K_0, IK_0) = g(K_0, K_0)i + g(K_0, KK_0)j - g(K_0, JK_0)k = v^{-1}i$, so that, by repeating this procedure also for J and K, we find

$$d\mu(IK_0) = v^{-1}i, \qquad d\mu(JK_0) = v^{-1}j, \qquad d\mu(KK_0) = v^{-1}k.$$

Then, by differentiating with respect to t the property of equivariance of μ at a point $m \in P$, $\mu(\exp(it)m) = \exp(it)\mu(m)\exp(-it) = i$, we find that $d\mu(Y_I) = 0$, and so $Y_I = \hat{K}_r + aK_0$, for some $a \in \mathbb{R}$. Analogously, from the relation $\mu(rm) = r\mu(m)r = r^2 i$, one obtains that $d\mu(Y_0) = 2i$, and so $Y_0 = \hat{Y} + bK_0 + 2vIK_0$, for some $b \in \mathbb{R}$. But then, by the formula $IY_0 = Y_I$, it follows that $a = -2v,\ b = 0$ and $Y = -IK_r$. Using the other relations (9), we eventually find

$$\begin{cases} Y_0 = -I\hat{K}_r - 2vIK_0 \\ Y_I = \hat{K}_r + 2vK_0 \\ Y_J = K\hat{K}_r + 2vKK_0 \\ Y_K = -J\hat{K}_r - 2vJK_0. \end{cases} \tag{11}$$

Another important consequence of the commutativity of the two S^1 actions on P is that the function v is S^1_r invariant. Indeed, differentiating with respect to t the commutativity relation $mf(t)z = mzf(t)$, where $z \in S^1_r$ and $f(t)$ is the integral curve starting at 1 of the generator of $\mathrm{Lie}(S^1_0) \cong \mathbb{R}$, one finds that $K_{0|zm} = d(L_z)K_{0|m}$ and so that

$$\begin{aligned} v_{zm}^{-1} &= g(K_{0|zm}, K_{0|zm}) = g(dL_z(K_{0|m}), dL_z(K_{0|m})) \\ &= (L_z^* g)(K_{0|m}, K_{0|m}) = g(K_{0|m}, K_{0|m}) = v_m^{-1}. \end{aligned}$$

For the same reason, also the connection ξ is S^1_r invariant, which means that $\mathcal{L}_{Y_I}\xi = 0$. But then, by Cartan's formula, we find that $0 = Y_I \lrcorner d\xi + Y_I \lrcorner d\xi$, and so that, denoting with $F_\xi \in \Omega^2(\tilde{M})$ the pullback to $\tilde{M}$ of the curvature of ξ,

$$K_r \lrcorner F_\xi + 2dv = 0, \tag{12}$$

which can be thought as an alternative formulation of the S^1_r invariance of ξ.

At this point, Haydys shows that the pullback metric f^*g on $\mathbb{H}^* \times P$ can be explicitly written in terms of the connection ξ on P, the function v and of tensors on $\mathbb{H}^*$ and $\tilde{M}$, so that also the metric on M_0 just depends on these quantities. We summarize here the procedure. Given $(x, m) \in \mathbb{H}^* \times P$, and (h_1, v_1) $(h_2, v_2) \in T_x\mathbb{H}^* \times T_m P$, define $\alpha = x^{-1}h_1,\ \beta = x^{-1}h_2 \in T_1\mathbb{H}^*$, and let Y_α, Y_β be the Killing vector fields of $\mathbb{H}^*$-action at m, corresponding to α and β respectively. Then one has $(Y_1, Y_i, Y_j, Y_k) = (Y_0, Y_I, Y_J, Y_K)$.
Recall now that $df_{(x,m)}(h_1, v_1) = \frac{d}{dt}_{|t=0} f(\gamma_1(t), \gamma_2(t))$, with γ_1 and γ_2 satisfying $\gamma_1(0) = x, \dot{\gamma}_1(0) = h_1$ and $\gamma_2(0) = m, \dot{\gamma}_2(0) = v_1$. But, since $h_1 = dL_x\alpha = \frac{d}{dt}_{|t=0}(x \cdot A(t))$, with A satisfying $A(0) = 1$ and $\dot{A}(0) = \alpha$, then

$$\begin{aligned} df_{(x,m)}(h_1, v_1) &= \frac{d}{dt}_{|t=0} xA(t)\gamma_2(t) = dL_x \left(\frac{d}{dt}_{|t=0} A(t)\gamma_2(t) \right) = \\ &= dL_x \left(\frac{d}{dt}_{|t=0} A(t) \cdot m + A(0) \cdot \frac{d}{dt}_{|t=0} \gamma_2(t) \right) \\ &= dL_x(Y_\alpha + v_1). \end{aligned}$$

Hence, one can write

$$f^*g((h_1, v_1), (h_2, v_2)) = g(dL_x(Y_\alpha + v_1), dL_x(Y_\beta + v_2)) = |x|^2 g(Y_\alpha + v_1, Y_\beta + v_2).$$

and now just the computations of the terms $g(Y_\alpha, Y_\beta), g(Y_\alpha, v)$ and $g(v_1, v_2)$ are needed. By developing them through (9) and 11, one obtain for the pullback metric the final form:

$$f^*g = (4v + w)\mathrm{Re}dx \otimes d\bar{x} - \mathrm{Re}(\bar{x}dxi \odot (2\xi + \eta)) + |x|^2(\tilde{g} + v^{-1}\xi^2), \tag{13}$$

where $(\delta \odot \gamma)(v_1, v_2) = \delta(v_1)\gamma(v_2) + \delta(v_2)\gamma(v_1)$ for 1-forms δ and γ, and η is now the pullback of η to P. Analogously,

$$f^*\omega((h_1, v_1), (h_2, v_2)) = \omega(dL_x(Y_\alpha + v_1), dL_x(Y_\beta + v_2)) = x\omega(Y_\alpha + v_1, Y_\beta + v_2)\bar{x}$$

and eventually

$$f^*\omega = \frac{4v + w}{2}\, dx \wedge d\bar{x} + x\tilde{\omega}\bar{x} - 2\mathrm{Im}(dxi\bar{x}) \wedge \xi - \mathrm{Im}(dxi \wedge \eta\bar{x}). \tag{14}$$

We can therefore state the following result:

Theorem 2.3. *If M is a hyperkähler manifold M with a permuting action of $\mathbb{H}^*$ and a tri-holomorphic action of S^1 with Killing field K_0 of squared norm v^{-1}, the open and everywhere dense submanifold $M_0 = M \setminus \mu^{-1}(0)$ can be seen as the product*

$$\mathbb{H}^* \times_{S^1_r} P,$$

where P is an S^1_0 principal bundle with connection ξ over the hyperkähler reduction $\tilde{M}$ of M and $S^1_r = Stab_i \subset \mathbb{H}^*$. Further, the hyperkähler structure on M_0 can be described just in terms of tensors on $\mathbb{H}^*$, $\tilde{M}$, and of ξ and v via (13) *and* (14).

Inverse construction The most important result of the section is that the previous construction can actually be reversed.

Start by considering a hyperkähler manifold $\tilde{M}$ endowed with an isometric and permuting complex structures S^1_r action. Take an S^1_0 principal bundle over $\tilde{M}$ with connection ξ and extend the S^1_r action on $\tilde{M}$ to P in such a way that the two S^1 actions on P commute (which is always possible, at least locally). Consider now the manifold $\mathcal{H}(\tilde{M}) = \mathbb{H}^* \times_{S^1_r} P$. One can define a metric g and a form ω on M_0 whose pull-backs to $\mathbb{H}^* \times P$ via $\pi : \mathbb{H}^* \times P \to \mathcal{H}(\tilde{M})$ are given by (13) and (14). For this to be true, one has to show that

1) the formulae (13) and (14) define invariant and basic tensors on $\mathbb{H}^* \times P$;

2) the 2-form $\omega \in \Omega^2(M_0, \mathrm{Im}\mathbb{H})$ is closed.

Now, (1) can be easily seen using the S^1_r-invariance of ξ and through a direct computation, so that (2) is equivalent to $f^*(\omega)$ being closed. Since $d\ f^*(\omega)$ lies in

$$\Omega^3(\mathbb{H}^* \times P, Im\mathbb{H}) = \bigoplus_{l=0}^{3} \Omega^l(\mathbb{H}^*, Im\mathbb{H}) \otimes \Omega^{3-l}(P, Im\mathbb{H}),$$

all its 4 components must vanish identically. This give rise to the following conditions:

$$F_\xi = -\frac{1}{2} d(k_r \lrcorner \tilde{g}) - \tilde{\omega}_I \tag{15}$$

and

$$4dv + dw = 2K_r \lrcorner \tilde{\omega}_I, \tag{16}$$

but, actually, we need just the first of them, since (16) follows from putting (15) and (12) together. The function v can be found, up to a constant, from (16).

Recall now that every hyperkähler manifold with an S^1 action preserving one complex structure and rotating the other two has a Kähler potential given by the moment map of the action itself. More precisely, if $\tilde{\rho} : \tilde{M} \to \mathbb{R}$ satisfies $d\tilde{\rho} = -K_r \lrcorner \tilde{\omega}_1$, then $j\partial_J \bar{\partial}_J \tilde{\rho} = \tilde{\omega}_J$ and the analogous holds for ω_K. Now, observing that $I^* d\tilde{\rho} = K_r \lrcorner \tilde{g}$ and so that $-2i\partial_I \bar{\partial}_I \tilde{\rho} = d(K_r \lrcorner \tilde{g})$, the equation (15) can be rewritten in the form

$$F_\xi = i\partial_I \bar{\partial}_I \tilde{\rho} - \tilde{\omega}_I$$

and $\tilde{\rho}$ is an hyperkähler potential when $F_\xi \equiv 0$.
We have therefore obtained the following result (the complete proof can be found in [10, Theorem 2.3]):

Theorem 2.4. *Let $\tilde{M}$ be a hyperkähler manifold endowed with an isometric and permuting complex structures S^1_r action. Let P an S^1_0 principal bundle over $\tilde{M}$ with connection ξ. Denoting by w the squared norm of the Killing vector field K_r of S^1_r and by $\tilde{\rho}$ its moment map, assume that the function*

$$v = -\frac{w + 2\tilde{\rho}}{4} \tag{17}$$

is everywhere positive.
Extend the S^1_r action on $\tilde{M}$ to P in such a way that the two S^1 actions on P commute. Then (13) *and* (14) *define a hyperkähler structure on $\mathcal{H}(\tilde{M}) = \mathbb{H}^* \times_{S^1_r} P$ if and only if*

$$F_\xi = i\partial_I \bar{\partial}_I \tilde{\rho} - \tilde{\omega}_I. \tag{18}$$

Note that, if one starts a hyperkähler manifold M with a permuting action of $\mathbb{H}^*$ and a triholomorphic action of S^1, the submanifold M_0 of the previous section can be obtained as $\mathcal{H}(\tilde{M})$, where $\tilde{M}$ is the hyperkähler reduction of M and satisfies the property above.

Note also that the hyperkähler reduction of $\mathcal{H}(\tilde{M})$ by S^1_0 is $\tilde{M}$, thus the construction above can be thought as an inverse to that of hyperkähler reduction.

One can say even more than Theorem 2.4 about $\mathcal{H}(\tilde{M})$. It was in fact proved by Feix that a hyperkähler potential for a hypkaähler manifold with a permuting $\mathbb{H}^*$ action is given by the squared norm of the Killing field of the action, up to a constant $-1/2$.
In our case, we have a natural permuting $\mathbb{H}^*$ action on $\mathcal{H}(\tilde{M})$ induced by the multiplication on the first component of $\mathbb{H}^* \times P$ and the squared norm of the Killing field of $\mathbb{R}^* \subset \mathbb{H}^*$ with respect to (13) is given by $\rho = (4v + w)|x|^2$. Hence we have the following

Corollary 2.5. *The left action of $\mathbb{H}^*$ action on $\mathcal{H}(\tilde{M})$ induces a transitive action on the 2-sphere of complex structures of $\mathcal{H}(\tilde{M})$, thus $\mathcal{H}(\tilde{M})$ has a hyperkähler potential given by*

$$\rho = -\frac{1}{2}(4v + w)|x|^2.$$

The associated quaternionic Kähler manifold So far, starting from a hyperkähler manifold $\tilde{M}$ with a specific S^1 symmetry, we have constructed another hyperkähler manifold $\mathcal{H}(\tilde{M})$ over it. Starting from $\mathcal{H}(\tilde{M})$, we can now construct the quaternionic Kähler manifold associated to $\tilde{M}$, by taking

$$\mathcal{Q}(\tilde{M}) = \mathcal{H}(\tilde{M})/\mathbb{H}^* = P/S^1_r,$$

which we know to be quaternionic Kähler by Theorem 2.1. We have passed in this way from the hyperkähler to the quaternionic Kähler side:

$$\begin{tikzcd}
 & & \mathbb{H} \times P \arrow[r] \arrow[d] & \mathcal{H}(\tilde{M}) \arrow[dd, bend left] & \mathbb{H}^* \arrow[l] \\
 & S^1_0 \arrow[r] & P \arrow[dr] & S^1_r \arrow[l] & \\
S^1_r \arrow[r] & \tilde{M} \arrow[ur] & & \mathcal{Q}(\tilde{M}) & S^1_0 \arrow[l]
\end{tikzcd}$$

The structure of $\mathcal{Q}(\tilde{M})$ can be described in terms of $\tilde{M}$ and P.
Take a level set of the hyperkähler potential of $\mathcal{H}(\tilde{M})$ and divide it by $Sp(1)$, so that the complex structures of N_0 can be viewed as induced by the ones on $\mathcal{H}(\tilde{M})$ on $V = \mathrm{span}(Y_0, Y_I, Y_J, Y_K)^\perp$.
Let $\lambda = (4v+w)^{-\frac{1}{2}}$; since $\rho_{|P} = -\frac{4v+w}{2}$, we have a diffeomorphism between P and the set $Q = \rho^{-1}(-\frac{1}{2}) \cap (\mu J + i\mu K)^{-1}(0) \cap \{\mu_I > 0\}$ given by

$$l : p \mapsto \lambda(p) \cdot p.$$

By denoting with ϖ the projection on V, we want now to compute the espressions

$$g_N = g(\varpi \circ dl\cdot, \varpi \circ dl\cdot) \text{ and } \omega(\varpi \circ dl\cdot, \varpi \circ dl\cdot).$$

Decomposing $u \in T_pP$ as $u_1 + u_2$, with $u_2 \in \mathrm{span}(Y_0, Y_I, Y_J, Y_K)$, and using relations (11), one can eventually find the espression for the metric

$$g_{N_0} = \frac{1}{4v+w}\left(\tilde{g} + \frac{1}{v}\xi^2 - \frac{1}{2(4v+w)}(2\xi+\bar{\eta}) \odot (2\xi+\eta)\right), \tag{19}$$

or, equivalently, denoting by $\varphi = \frac{2\xi + K_r \lrcorner \tilde{g}}{4v+w}$,

$$g_{N_0} = \frac{1}{4v+w}\left(\tilde{g} + \frac{1}{v}\xi^2 - \frac{1}{2}\varphi^2\right) - \frac{(\mathrm{Im}\eta)^2}{2(4v+w)}, \tag{20}$$

which is well defined as metric on N_0.
In a similar way, one can find that the fundamental 4-form on N_0 is given by

$$\Omega = \chi_I \wedge \chi_I + \chi_J \wedge \chi_J + \chi_K \wedge \chi_K, \tag{21}$$

where χ_I, χ_J, and χ_K are the i, j, k components of

$$\chi = \frac{1}{4v+w}\tilde{\omega} - \frac{1}{2(4v+w)}(2\xi+\bar{\eta}) \wedge (2\xi+\eta).$$

Theorem 2.6. *If the hyphotesis of 2.4 are satisfied, then* (19) *and* (21) *define a quaternionic Kahler structure on the manifold $\mathcal{Q}(\tilde{M})$, which admits a quaternionic Kahler action of S^1.*

Symmetries of the construction We want now to look briefly at some symmmetries and asymmetries of the presented construction.

Observe fist that, if we start from an hyperkähler manifold $\tilde{M}$ and we pass to the quaternionic Kähler manifold $\mathcal{Q}(\tilde{M})$ through $\mathcal{H}(\tilde{M})$, the Swann bundle $\mathcal{U}(\mathcal{Q}(\tilde{M}))$ is exactly $\mathcal{H}(\tilde{M})$.

On the other hand, let us take a quaternionic Kähler manifold N and perform the construction to find the corresponding hyperkähler manifold $\mathcal{K}(N)$. If we now come back to the quaternionic Kähler side starting from $\mathcal{K}(N)$, what we find is not N, but instaed the open and everywhere dense manifold $N_0 = N \setminus \{\phi = 0\}$, where

$$\phi : N \to \mathcal{A}$$

is the quaternionic Kähler moment map associated to the S^1_0 action on N.

$$\begin{tikzcd}
& \mathcal{H}(\mathcal{K}(N)) \arrow[dr, dotted] & \\
\mathcal{K}(N) \arrow[ur] & & N \arrow[dl] \\
& \mathcal{U}(N) \arrow[ul] &
\end{tikzcd}$$

Indeed, when taking the quotient P/S^1_r, we are away from the level set $\{\mu = 0\}$, where here μ denotes the hyperkähler moment map for the action of S^1 on $\mathcal{U}(N)$, and, from the quotient constructions for quaternionic Kähler manifolds seen in Section 1, we can observe that the set $\{\mu = 0\} \subset \mathcal{U}(N)$ corresponds exactly to the set $\{\phi = 0\} \subset N$.

2.3.2 The approach via twistor theory

In [11] Hitchin describes a way to study the correspondence from the point of view of twistor spaces. See, for a treatement of twistor theory, [2]. Here we just recall that the twistor space $\hat{Z}$ of a quaternionic Kähler manifold is a $(2n+1)$-dimensional Kähler-Einsten manifold with a complex contact structure, given by a holomorphic section $\alpha \in T^*_{\hat{Z}} \otimes K_{\hat{Z}}^{-1/(k+1)}$, where T^* denotes the cotangent bundle, while K the anticanonical line bundle of $\hat{Z}$.
The idea of the construction then goes as follows.

We have seen that, to pass from the hyperkähler to the quaternionic Kähler side of the correspondence, one needs the introduction of a S^1 principal bundle P over the manifold $\tilde{M}$. Given that, one just lifts the S^1 circle action on $\tilde{M}$ to P and take the quotient manifold.

Haydys introduces the bundle via its curvature form $dd^c_I\rho - \tilde{\omega}_I$, where $d^c_I = I^* dI$, ω_I is the invariant Kähler form and ρ the relative moment map for the S^1 action on $\tilde{M}$.

Hitchin's idea, instead, starts from the observation that the bundle considered is hyperholomorphic: indeed, one can prove that $dd^c_I\rho - \tilde{\omega}_I$ is of type $(1,1)$ with respect to all the three complex structures I, J, K, and so, with respect to any complex structure of the 2-sphere family of complex structures on $\tilde{M}$, thus (if the cohomology class $\frac{\omega}{2\pi}$ is integral) it can be thought as curvature of a hyperholomorphic connection.

At this point, he shows that a hyperholomorphic line bundle on $\tilde{M}$ is uniquely determined by a holomorphic line bundle L_Z on the twistor space Z of $\tilde{M}$ which, further, depends only on the geometry of the twistor space and not on the twistor lines (unlike the hyperkähler metric on $\tilde{M}$).

The holomorphic line bundle L_Z on Z Since a hyperholomorphic connection has curvature of type $(1,1)$ with respect to all $I_x = x_1 I + x_2 J + x_3 K$, with $(x_1, x_2, x_3) \in S^2$, it follows that such a connection on $\tilde{M}$, pulled back to $Z = \tilde{M} \times S^2$ defines a holomorphic structure on the bundle. The definition of the hyperholomorphic line bundle on $\tilde{M}$ through twistors hence consists in looking for a holomorphic line bundle L_Z on the twistor space.
The starting point is the possibility to define a natural holomorphic line bundle L_Z on the twistor space of a hyperkähler manifold with a circle action, which is trivial on every twistor line and corresponds to a hyperholomorphic line bundle L on $\tilde{M}$; the description of such an object involves the use of *Čech cohomology* (see [11, Section 2]), but one can think at it by observing that L_Z on $Z \to \mathbb{P}^1$ has a meromorphic connection with poles on the fibres over 0 and ∞. The curvature of such a connection is a closed meromorphic 2-form that, restricted to each fibre over $\mathbb{P}^1 \setminus \{0, \infty\}$, gives a multiple of the holomorphic symplectic form ω.
Indeed, Hitchin then proves that L_Z coincides with the holomorphic line bundle P with curvature $F = dd^c_I \rho - \tilde{\omega}_I$ on $Z = \tilde{M} \times S^2$. (see ([11, Section 3]).

Now one can pass from the twistor space of the hyperkähler manifold to the one of the corresponding quaternionic Kähler manifold: the contact form for the quaternionic Kähler metric will be defined in terms of the meromorphic connection on L_Z.

The quaternionic Kähler manifold On the twistor space Z of a hyperkähler manifold with a circle action we have now a principal $\mathbb{C}^*$ bundle P with a lifted action. Assume for convenience that the holomorphic vector field Y associated to the circle action generates a $\mathbb{C}^*$ action on P; then, after removing the fixed points, we can define $\hat{Z}$ as the quotient.

$$\begin{tikzcd}
 & \mathbb{C}^* \arrow[r] & P \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \mathbb{C}^*_r \arrow[l] & \\
 & Z \arrow[d] & & \hat{Z} \arrow[d] & \\
S^1_r \arrow[r] & \tilde{M} & & N & S^1_0 \arrow[l]
\end{tikzcd}$$

Hitchin shows that $\hat{Z}$ has a contact structure α invariant by the induced $\mathbb{C}^*$ action and is therefore a holomorphic contact manifold: indeed, if $\hat{p} : P \to \hat{Z}$ is the quotient map for the action, since the tangent bundle of P along the fibres is trivial, $K_P = \hat{p}^* K_{\hat{Z}}$, and similarly, $K_P = p^* K_Z = \mathcal{O}(-2(k+1))$ by the properties of the twistor space Z of $\tilde{M}$, so

$$\hat{p}^* K_{\hat{Z}}^{-1/(k+1)} \cong p^* \mathcal{O}(2).$$

We should then define our contact form α on $\hat{Z}$ as a section of $T^*_{\hat{Z}}(2)$.
This is done in terms of the connection on P : consider the meromorphic connection 1-form A on P with poles at $\xi = 0, \infty$. This defines a section $\xi A \in T^*_P(2)$, which is clearly invariant under the circle action and, further, results to be non-degenerate and such that the contraction $\tilde{Y} \lrcorner A$ vanishes identically. These are exactly the properties that $\hat{p}^*\alpha \in T^*_P(2)$ should verify, so we can choose our desired form α to be defined by $\hat{p}^*\alpha = \xi A$.
Since then the bundle P is trivial on the twistor lines of Z, they lift to P and descend to $\hat{Z}$ to define the twistor lines on $\hat{Z}$. The space $\hat{Z}$ is then completely determined and is the twistor space of the associated quaternionic Kähler manifold.

3 Examples

This section is devoted to the attempt to understand how the correspondence between hyperkähler and quaternionic Kähler manifolds admitting S^1 symmetries work in some explicit examples.

3.1 The cotangent space $T^*\mathbb{CP}^n$

Consider as a first example the manifold $T^*\mathbb{CP}^n$.
We have seen in section 1 that the hyperk ähler quotient of $\mathbb{H}^{n+1}$ by S^1 acting by multiplication on the left with respect to a non-zero value of the moment map is topologically $T^*\mathbb{CP}^n$ and that the metric coincides with the one defined by Calabi.
Hence, we have that in this case that $\mathcal{H}(T^*\mathbb{CP}^n) = \mathbb{H}^{n+1^*}$ with its flat metric (we need to remove the zero level set of the relative moment map). Consequently, $\mathcal{Q}(T^*\mathbb{CP}^n) = \mathbb{H}^{n+1^*}/\mathbb{H}^*$ is $\mathbb{HP}^n$, but also here we need to remove the zero set of the corresponding moment map.
Recall that we have previously seen that the Swann bundle $\mathcal{U}(\mathbb{HP}^n)$ is in fact $\mathbb{H}^{n+1^*}$.

3.2 Flat manifolds

Consider as manifold $\tilde{M}$ the ring $\mathbb{H}$ of quaternions, with coordinate $q = q_0 + q_1 i + q_2 j + q_3 k$, and the following action of S^1_r :

$$(z, q) \mapsto zq\bar{z}$$

Using the notations from the previous chapter and choosing as triple of symplectic forms the self-dual one, it is easy to see that, in this case, $K_r(q) = -2q_3 j + 2q_2 k$ and so $w = 4(q_3^2 + q_2^2)$ and, further, we have

$$d\tilde{\rho} = -K_r \lrcorner \tilde{\omega}_1 = -2q_3 dq_3 - 2q_2 dq_2 = d(-q_3^2 - q_2^2).$$

Then $-\frac{w+2\tilde{\rho}}{4} = -\frac{q_3^2+q_2^2}{2}$ and we may choose as function v

$$v = \frac{1}{2}(1 - q_3^2 - q_2^2),$$

which is positive on $\mathbb{R}^2_{y_0y_1} \times D^2_{y_2y_3}$ ($D^2 \subset \mathbb{R}^2$ is the open disk of radius 1). Then, the principal S^1_0-bundle P is trivial and formulae (13) and (14) define an hyperkähler structure on $M_0 = \mathcal{H}(\mathbb{R}^2 \times D) = \mathbb{H}^* \times \mathbb{R}^2 \times D$ if and only if

$$
\begin{aligned}
F_\xi &= i\partial_1\bar{\partial}_1\tilde{\rho} - \tilde{\omega}_1 = -\frac{1}{2}d(k_r \lrcorner \tilde{g}) - \tilde{\omega}_1 \\
&= 2dq_3 \wedge dq_2 + dq_0 \wedge dq_1 + dq_2 \wedge dq_3 = dq_0 \wedge dq_1 + dq_3 \wedge dq_2,
\end{aligned}
$$

so so that $\xi = q_0 dq_1 + q_3 dq_2$. Moreover, $\mathcal{Q}(\mathbb{R}^2 \times D) = \mathbb{R}^2 \times D$ and, setting $d = q_3^2 + q_2^2$, one finds that

$$
\begin{aligned}
g_N &= \frac{1}{2(1+d)}(\tilde{g} + \frac{2}{1-d}\xi^2 - \frac{1}{4(1+d)}(8\xi^2 + 4\xi \odot \mathrm{Re}\eta + \eta \odot \bar{\eta})) \\
&= \frac{1}{2(1+d)}(\tilde{g} + \frac{4d}{(1-d)^2}\xi^2 - \frac{1}{4(1+d)}(4\xi \odot (k_r \lrcorner \tilde{g}) + 2\mathrm{Re}(\eta \otimes \bar{\eta}))).
\end{aligned}
$$

Now, since

$$
\begin{aligned}
\mathrm{Re}(\eta \otimes \bar{\eta})) &= (k_r \lrcorner \tilde{g})^2 + (k_r \lrcorner \tilde{\omega}_1)^2 + (k_r \lrcorner \tilde{\omega}_2)^2 + (k_r \lrcorner \tilde{\omega}_3)^2 \\
&= 4((q_2 dq_3 - q_3 dq_2)^2 + (q_2 dq_2 + q_3 dq_3)^2 \\
&\quad + (q_3 dq_0 - q_2 dq_1)^2 + (q_2 dq_0 - q_3 dq_1)^2) \\
&= 4d\ (dq_0^2 + dq_1^2 + dq_2^2 + dq_3^2) \\
&= 4d\ \mathrm{Re}\ dq \otimes d\bar{q},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
g_N &= \frac{1}{2(1+d)}(\frac{1-d}{1+d}\ \mathrm{Re}\ dq \otimes d\bar{q} + \frac{4d}{(1-d)^2}\xi^2 - \frac{1}{(1+d)}(\xi \odot k_r \lrcorner \tilde{g})) \\
&= \frac{1}{2(1+d)}\Big(\frac{1-d}{1+d}\ \mathrm{Re} dq \otimes d\bar{q} + \frac{4d}{(1-d)^2}(q_0 dq_1 + q_3 dq_2)^2 \\
&\quad - \frac{1}{(1+d)}(q_0 dq_1 + q_3 dq_2) \odot (q_2 dq_3 - q_3 dq_2)\Big).
\end{aligned}
$$

The metric g_N is Einsten and antiself-dual but incomplete. Similarly, one can compute the metric on $\mathbb{H}^* \times \mathbb{R}^2 \times D$, which is also incomplete.

Notice that, choosing as triple of symplectic forms the antiselfdual one, we would have found $d\tilde{\rho} 2q_3 dq_3 + 2q_2 dq_2 = d(q_3^2 + q_2^2)$, thus $-\frac{w+2\tilde{\rho}}{4} = -\frac{3}{2}(q_3^2 + q_2^2)$ and we may choose the function v as $v = \frac{3}{2}(1 - q_3^2 - q_2^2)$, positive on the set $\mathbb{R}^2_{y_0y_1} \times D^2_{y_2y_3}$. In this case, one has a hyperkähler structure on $M_0 = \mathcal{H}(\mathbb{R}^2 \times D) = \mathbb{H}^* \times \mathbb{R}^2 \times D$ if and only if

$$
F_\xi = 2dq_2 \wedge dq_3 + dq_0 \wedge dq_1 + dq_3 \wedge dq_2 = dq_0 \wedge dq_1 + dq_2 \wedge dq_3
$$

so that $\xi = q_0 dq_1 + q_2 dq_3$.

3.3 Gibbons-Hawking Spaces

We have seen that, if $\tilde{M}$ is hyperkähler with a S^1 triholomorphic action generated by the vector field W, then the hyperkähler moment map shows $\tilde{M}$ as a fibration over an open set in R^3 with generic fiber S^1.
In general such manifolds do not admit an S^1_r action. However, if we choose the function V satysfying (1) of the form (7), one can see that, when all the poles u_j of V lie on the same line, then such action exists and its projection to $\mathbb{R}^3$ is given by $(z, x) \to zx\bar{z}$, with $x \in \mathbb{H}$.

Taub-NUT metrics Consider the Taub-NUT metric as in Section 1.1.2. Recall that, in this case, our hyperkähler manifold $\tilde{M}$ is still the space $\mathbb{H}$, endowed in addition with an S^1 action given by $e^{it} \cdot q = e^{it}(q_0+q_1 i)+e^{it}(q_0+q_3 i)j$ and the moment map $\mu : \mathbb{H} \to \mathrm{Im}\mathbb{H} \cong \mathbb{R}\oplus\mathbb{C}$ associated to it, with respect to the anti-self dual basis, is given by $\mu(q) = \bar{q}iq = u_1 i + k(u_2 + iu_3)$, with

$$\begin{cases} u_1 = q_0^2 + q_1^2 - q_2^2 - q_3^2 \\ u_2 = 2(q_0 q_2 + q_1 q_3) \\ u_3 = 2(q_1 q_2 - q_0 q_3). \end{cases}$$

The Taub-NUT metric $\tilde{g} = V_e \; du \cdot du + \frac{1}{V_e}\theta_0^2$ on $\mathbb{H}$ $(e > 0)$, with

$$\theta_0 = \frac{1}{2\pi}\frac{-q_1 dq_0 + q_0 dq_1 - q_3 dq2 + q_2 dq_3}{q_0^2 + q_1^2 + q_2^2 + q_3^2}$$

and $V_e = e + \frac{1}{4\pi|u|} = e + \frac{1}{4\pi|q|^2}$ is hyperkähler with respect to three 2-forms (2) and becomes, in terms of the q_i,

$$\begin{aligned} \tilde{g} = & \frac{1}{\pi|u|}[(4\pi|u|e+1)(q_0^2+q_2^2+q_3^2) + \frac{1}{4\pi|u|e+1}q_1^2]dq_0^2 \\ & + \frac{1}{\pi|u|}[(4\pi|u|e+1)(q_1^2+q_2^2+q_3^2) + \frac{1}{4\pi|u|e+1}q_0^2]dq_1^2 \\ & + \frac{1}{\pi|u|}[(4\pi|u|e+1)(q_0^2+q_1^2+q_2^2) + \frac{1}{4\pi|u|e+1}q_3^2]dq_2^2 \\ & + \frac{1}{\pi|u|}[(4\pi|u|e+1)(q_0^2+q_1^2+q_3^2) + \frac{1}{4\pi|u|e+1}q_2^2]dq_3^2 \\ & + \frac{8e(2\pi|u|e+1)}{4\pi|u|e+1}[q_o q_1 dq_o \odot dq_1 - q_3 q_1 dq_o \odot dq_2 + q_2 q_1 dq_o \odot dq_3 \\ & + q_3 q_0 dq_1 \odot dq_2 - q_0 q_2 dq_1 \odot dq_3 + q_2 q_3 dq_2 \odot dq_3] \end{aligned} \tag{22}$$

The $S^1_r = Stab_i$-action on $\mathbb{H}$ must commute with the projected one:

$$(\overline{z \cdot q})i(z \cdot q) = \mu(z \cdot q) = z\mu(q)\bar{z} = z\bar{q}iq\bar{z} = \overline{zq}izq,$$

thus one has $z \cdot q = zq\bar{z}$. It is easily verified that $(L_z)^*(\omega) = z\omega\bar{z}$. We still have that $k_r = [i, q] = -2q_3 j + 2q_2 k$, but its squared norm is now

$$\begin{aligned} w &= \frac{4}{\pi|u|}\frac{1}{4\pi|u|e+1}(q_2^2+q_3^2)^2 + \frac{4}{\pi|u|}(4\pi|u|e+1)(q_2^2+q_3^2)(q_0^2+q_1^2) \\ &= \frac{4}{\pi}(q_2^2+q_3^2)\frac{1+8\pi e(2\pi|u|e+1)(q_0^2+q_1^2)}{4\pi|u|e+1} \end{aligned} \tag{23}$$

Since

$$\begin{aligned} \tilde{\omega}_1 &= du_1 \wedge \theta_0 + V du_2 \wedge du_3 \\ &= 4e[(q_3q_0 + q_1q_2)dq_{02} + (q_3q_1 - q_0q_2)dq_{03} \\ &+ (q_3q_1 - q_0q_2)dq_{12} - (q_3q_0 + q_1q_2)dq_{13}] \\ &+ [\frac{1}{\pi} + 4e(q_2^2+q_3^2)]dq_{01} + [\frac{1}{\pi} + 4e(q_0^2+q_1^2)]dq_{32} \end{aligned}$$

we have $d\tilde{\rho} = K_r \lrcorner \tilde{\omega}_1 = d[4e(q_0^2+q_1^2) + \frac{1}{\pi}(q_2^2+q_3^2)]$, thus the moment map $\tilde{\rho}$ is $\tilde{\rho} = 4e(q_0^2 + q_1^2) + \frac{1}{\pi}(q_2^2+q_3^2)$. Further, one finds

$$v = -\frac{3}{2\pi}\frac{q_2^2+q_3^2}{4\pi|u|e+1}[8\pi e(2\pi|u|e+1)(q_0^2+q_1^2) + 1 + \frac{4}{3}\pi e(q_2^2+q_3^2)]$$

so, to have the flat case as degenerate one for $e = 0$, we may choose

$$v + \frac{3}{2\pi} = \frac{3}{2\pi}\{-\frac{q_2^2+q_3^2}{4\pi|u|e+1}[8\pi e(2\pi|u|e+1)(q_0^2+q_1^2) + 1 + \frac{4}{3}\pi e(q_2^2+q_3^2)] + 1\}$$

As seen above, for $e = 0$, the set where this function is positive is given by $\mathbb{R}^2_{y_0y_1} \times D^2_{y_2y_3}$. For $e > 0$ we get a perturbation $\mathcal{D}$ of it that is still a cylinder topologically (for $|u| \to \infty$, this is the interior of a hyperboloid in $\mathbb{C}^2$).
Now, the curvature for the S^1_0-bundle P over $\mathbb{H}$ is given by the formula (16), thus, in order to find the connection ξ, one just needs to compute $K_r \lrcorner \tilde{g}$ and to find a 1-form α such that $d\alpha = \tilde{\omega}_1$, so that (up to closed forms) $\xi = +\frac{1}{2}(K_r \lrcorner \tilde{g}) + \alpha$. By denoting by $h = 4\pi|u|e + 1$ and $g = \frac{2\pi|u|e+1}{4\pi|u|e+1}$, we find

$$K_r \lrcorner \tilde{g} = (16e \cdot g(q_0^2+q_1^2) + \frac{2}{\pi h})(-q_3dq_2 + q_2dq_3) + 16e \cdot g(q_2^2+q_3^2)(q_1dq_0 - q_0dq_1).$$

To find α, write $\alpha = \alpha_0dq_0 + \alpha_1dq_1 + \alpha_2dq_2 + \alpha_3dq_3$ and impose the following system of PDEs

$$\begin{cases} \frac{1}{\pi} + 4e(q_2^2+q_3^2) = -\frac{\partial\alpha_0}{\partial q_1} + \frac{\partial\alpha_1}{\partial q_0} \\ \frac{1}{\pi} + 4e(q_0^2+q_1^2) = +\frac{\partial\alpha_2}{\partial q_3} - \frac{\partial\alpha_3}{\partial q_2} \\ 4e(q_3q_0 + q_1q_2) = -\frac{\partial\alpha_0}{\partial q_2} + \frac{\partial\alpha_2}{\partial q_0} \\ 4e(q_3q_1 - q_0q_2) = -\frac{\partial\alpha_0}{\partial q_3} + \frac{\partial\alpha_3}{\partial q_0} \\ 4e(q_3q_1 - q_0q_2) = -\frac{\partial\alpha_1}{\partial q_2} + \frac{\partial\alpha_2}{\partial q_1} \\ -4e(q_3q_0 + q_1q_2) = -\frac{\partial\alpha_1}{\partial q_3} + \frac{\partial\alpha_3}{\partial q_1} \end{cases}$$

A solution of this system is

$$\begin{cases} \alpha_0 = 4e(-\frac{1}{2\pi}q_3^2 q_1 - \frac{1}{2\pi}q_2^2 q_1) - \frac{1}{2\pi}q_1 \\ \alpha_1 = 4e(\frac{1}{2\pi}q_2^2 q_0 + \frac{1}{2\pi}q_3^2 q_0) + \frac{1}{2\pi}q_0 \\ \alpha_2 = 4e(\frac{1}{2\pi}q_1^2 q_3 + \frac{1}{2\pi}q_0^2 q_3) + \frac{1}{2\pi}q_3 \\ \alpha_3 = 4e(-\frac{1}{2\pi}q_1^2 q_2 - \frac{1}{2\pi}q_0^2 q_2) - \frac{1}{2\pi}q_2 \end{cases}$$

Then a possible ξ is given by the coefficients

$$\begin{aligned} \xi_0 &= \frac{16eq_1}{2} g(q_2^2 + q_3^2) - \frac{2eq_1}{\pi}(q_2^2 + q_3^2 + \frac{1}{4e}) \\ \xi_2 &= -\frac{q_3}{\pi|u|}[(h(q_0^2 + q_1^2) + \frac{1}{h}(q_2^2 + q_3^2)] + \frac{2eq_3}{\pi}[(q_0^2 + q_1^2) + \frac{1}{4e}], \end{aligned} \tag{24}$$

while $\xi_1 = -\xi_0$ and $\xi_3 = -\xi_2$. We add to this 1-form the closed form

$$\frac{q_1}{2\pi}dq_0 + \frac{q_0}{2\pi}dq_1 + \frac{q_3}{2\pi}dq_2 + \frac{q_3}{2\pi}dq_3$$

in order to make our choice of ξ coincide with the one of the flat case for $e = 0$. The S^1_0-bundle P is now completely determined. Substituing $\tilde{g}, \xi, k_r \lrcorner \tilde{g}, k_r \lrcorner \tilde{\omega}_i$ in formula (19), we find then a family $g_{N,e}$ of Einsten self-dual metrics on degenerations of $\mathcal{H}(\mathbb{R}^2_{y_0 y_1} \times D^2_{y_2 y_3})$ which are incomplete.
Notice that the hyperkähler-quaternionic Kähler correspondence in this case (and in the flat one) destroys all the symmetries of the original space.

3.4 The Eguchi-Hanson case

We now describe as the studied correspondence can be understood for the case of Eguchi-Hanson metric on T^*S^2 via the twistor approach, as done by Hitchin in [11, Section 4.4].
Consider then as hyperkähler manifold with a circle action the space $T^*S^2 \cong T^*\mathbb{CP}^1$, with the natural action on fibres. Recall that the metric considered can be found also as hyperkähler quotient of $\mathbb{H}^2$ by circle action; further, the twistor space Z can be seen as the quotient of an open set inside

$$W = \{(v, \xi) \in V(1) \oplus V^*(1) \to \mathbb{P}^1 : \langle v, \xi \rangle = \zeta\}, \tag{25}$$

with V a 2-dimensional vector space, via the action $(v, \xi) \to (\lambda v, \lambda^{-1}\xi)$, and, through considerations of $U(2)$ invariance, W, with the $\mathbb{C}^*$ action, turns out to be the holomorphic principal bundle P for Eguchi-Hanson.
Consider the lift $(v, \xi) \to (\nu v, \nu \xi, \nu^2 \zeta)$ of the geometrical action on M. The equality $\langle v, \xi \rangle = \zeta$ determines ζ in terms of v and ξ. Hence, the quotient is $\mathbb{P}(V \oplus V^*) \cong \mathbb{P}^3$, which is the twistor space of S^4. Then, $\mathcal{Q}(T^*S^2) = S^4$ and the quaternionic Kähler metric is the standard metric on the sphere.
Hitchin underlines that, more precisely, the associated quaternionic Kähler manifold is the complement of a circle in S^4, since one has to remove fixed points of the circle action on the hyperkähler side. Moreover, there is also a choice in lifting the geometric action on the twistor space.

3.5 The projective space $\mathbb{CP}^2$?

We now start from the quaternionic Kähler side, by considering the quaternionic Kähler manifold $\mathbb{CP}^2$. We wonder what the corresponding hyperkähler manifold is.
The first step is to determine the Swann bundle of $\mathbb{CP}^2$. This is, locally, the negative spin bundle $V_- \setminus 0$ over $\mathbb{CP}^2$ (to have it well defined globally, one has to quotient by $\mathbb{Z}_2$ since $\mathbb{CP}^2$ is not a spin manifold), as explained by Swann in [13, Chapter 2].
Notice that $\mathbb{CP}^2$ is the quaternionic Kähler quotient of $\mathbb{HP}^2$ by the quaternionic kähler action of S^1 defined in the first section. Thus, one can also obtain the Swann bundle $\mathbb{CP}^2$ as the hyperkähler reduction, at level 0, of $\mathbb{HP}^{\in} = \mathbb{H}^3\ 0$ by the lifted S^1 action.

$$\begin{tikzcd}
S^1_{(hk)} \arrow[r, bend left] & \mathbb{H}^3\ 0 \arrow[r, "(hk)"] \arrow[d] & V_- \arrow[d] \\
S^1_{(qk)} \arrow[r, bend left] & \mathbb{HP}^2 \arrow[r, "(qk)"] & \mathbb{CP}^2
\end{tikzcd}$$

One should now define a quaternionic Kähler circle action on $\mathbb{CP}^2$, lift it to a hyperkähler circle action on negative spin bundle V_-, and perform the hyperkähler reduction.
One may try to do that by taking a $U(1)$ inside SU(3) acting on $\mathbb{CP}^2$, and lifting it to the spin bundle.

3.6 Considerations

It is worth to underline that, from the considered examples, one can see how the studied correspondence, acting from the hyperkähler to the quaternionic Kähler side, can destroy all the symmetries of the underlying original space: this is true for the Taub-NUT metric and already for the flat case.
We leave to further investigation the case of the projective space $\mathbb{CP}^2$. A way to start could be to anaylize the possible triholomorphic circle actions of the negative spin bundle V_- over $\mathbb{CP}^2$ and the relative hyperkahler quotients.

4 Construction of twistor spaces in dimension 4

Given an oriented Riemannian 4-manifold, (M, g, θ), where g is the metric and θ is the volume form, we want to study the space of almost complex structures, J which are compatible with both the metric and the orientation in the sense that $g(JX, JY) = g(X, Y)$ and $\{e_1, Je_1, e_2, Je_2\}$ is an oriented basis for any $e_1, e_2 \in T_mM$. By defining a 2-form, ω by $\omega(X, Y) = g(JX, Y)$ the second condition can be stated as $\omega \wedge \omega = 2\theta$ and the first condition is replaced by demanding that ω has unit length with respect to g.

$SO(4)$ has a natural representation on $\mathbb{R}^4$, hence this induces a representation on $\Lambda^2(\mathbb{R}^4)^* \cong \mathfrak{so}(4)$. This representation is reducible and decomposes the space of 2-forms into the self-dual (SD) and anti-self dual (ASD) ones spanned by;

$$\begin{aligned}
\omega^1_\pm &= dx_{12} \pm dx_{34}\\
\omega^2_\pm &= dx_{13} \pm dx_{42}\\
\omega^3_\pm &= dx_{14} \pm dx_{23}
\end{aligned}$$

in accordance to the splitting of the adjoint representation, $\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3)$. The unit 2-forms compatible with θ are the SD ones. Since $\mathfrak{so}(3) \cong \mathbb{R}^3$, the orbit of any given non-zero element in $\Lambda^2(\mathbb{R}^4)^*$ is a 2-sphere. Put differently, the stabiliser of any non-degenerate 2-form in $SO(4)$ is $U(2)$, hence the orbit space is the symmetric space $SO(4)/U(2)$ which is diffeomorphic to S^2. This sphere parametrises the space of almost complex structures on $\mathbb{R}^4$. This analysis is transferred to M by the identification $\mathbb{R}^4 \cong T_mM$ defined by any compatible frame at m. If we denote the reduced oriented orthonormal frame bundle of M by P then we can define the associated bundle of ASD 2 forms by

$$\Lambda^2_- M = P \times_{SO(4)} \Lambda^2_-(\mathbb{R}^4)^*.$$

The (orientation reversing) twistor space, $\pi : Z \to M$ is then defined to be the unit sphere bundle in $\Lambda^2_- M$. By orientation reversing, we mean here that we are working with the sphere bundle in the ASD bundle instead of the SD one. Then an almost complex structure on M is simply a section of this bundle (after pairing with the metric). We note here that the almost complex structure is independent of the choice of the metric in any conformal class since if we replace g by $e^{2f}g$ for some function on M then θ is replaced by $e^{4f}\theta$, hence J is unchanged.

The vector bundle, $\Lambda^2_- M$ inherits the Levi-Civita connection from P and since the connection preserves the metric it follows that horizontal curves in Z remain in Z. So the tangent bundle of Z splits into a horizontal part, H and vertical part, V (the tangent to the sphere fibres). At the point $z = (m, \hat{j}) \in Z$, where $\hat{j}$ is an almost complex structure on T_mM i.e. a point in the fibre, $\pi^{-1}(m)$, there is a natural almost complex structure acting on H_z by

$$\begin{aligned}
J^H_z : H_z &\to H_z\\
X &\mapsto \pi_*^{-1}\hat{j}(\pi_* X)
\end{aligned}$$

The fibres are $\mathbb{P}^1$s and so have a natural almost complex structure, $J_z^V : V_z \to V_z$ (compatible with the orientation given by the normal pointing outwards of the sphere) but there is also another choice of almost complex structure on V namely $-J_z^V$. Combining J^H with either of these two, we have 2 natural choices of almost complex structures on Z which we denote by J and $\hat{J}$ respectively. The natural question that now arises is when, if ever, are any of these almost complex structures integrable. This is answered by the following two well-known results;

Theorem 4.1 ([1])**.** *Z is a complex manifold with respect to J iff M is a self-dual manifold i.e. ASD Weyl tensor vanishes.*

Theorem 4.2 ([4])**.** *Z is never a complex manifold with respect to $\hat{J}$.*

Two fundamental features that Z possesses are a fixed point free antiholomorphic involution which preserves the fibres in the sense that it maps each $\mathbb{P}^1$ fibre to itself and the normal bundle of each fibre is biholomorphic to the bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. The fundamental theorem of Penrose is that these are in fact sufficient data to recover M.

Theorem 4.3. *[8] If a complex 3-fold Z has a $\mathbb{P}^1$ fibration whose normal bundles are $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and also has an anti-holomorphic fibre preserving involution then Z is the twistor space of some SD 4-manifold M.*

The two well-known examples of twistor maps are $\mathbb{P}^3 \to S^4$ (Penrose fibration) and $\mathbb{F}^{(1,2,3)} \to \mathbb{P}^2$. The first one can be described very simply as follows;

We identify $\mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{H}^2$ via the map $((x, y), (u, v)) \mapsto (x + jy, u + jv)$ where j is the unit quaternion in $\mathbb{H}$, and $S^4 = \mathbb{HP}^1$ by the stereographic projection. We consider $\mathbb{C}$ as a subalgebra of $\mathbb{H}$ which itself acts by right multiplication on $\mathbb{H}^2$. The projection of the complex line in $\mathbb{P}^3$ to its quaternionic line spanned in $\mathbb{H}^2$ (over $\mathbb{H}$) is the Penrose fibration, explicitly

$$[x : y : u : v]_{\mathbb{C}} \mapsto [x + jy : u + jv]_{\mathbb{H}}.$$

The fibre $\pi^{-1}[1 : 0] = \{[a : b : 0 : 0] \mid (a, b) \in \mathbb{C}^2 - (0, 0)\} \cong \mathbb{P}^1$, as expected.

5 Twistor spaces in higher dimensions

Before moving to the description of twistor spaces in higher dimensions, which one can hope to achieve only in dimensions which are a multiple of 4, we recall that the definition of quaternionic Kähler manifolds means that the holonomy group is contained in $Sp(n)Sp(1)$. In dimension 4, this definition would be trivially satisfied for any oriented Riemannian manifold since $Sp(1)Sp(1) = SO(4)$. So we need more to define an analogue of QK structure in dimension 4. Two properties of QK manifolds are that they are Einstein and moreover the curvature operator, $R : \Lambda^2 TM^* \to \Lambda^2 TM^*$, $e^{ij} \mapsto R_{ijkl}e^{kl}$ where e^i is an orthonormal basis, restricted to the bundle, $\mathcal{A}$ defined earlier is a positive multiple of the identity map. With respect to the decomposition of 2-forms as SD and ASD ones, R admits the block decomposition

$$R = \begin{pmatrix} A & B \\ B* & D \end{pmatrix} : \Lambda^2_+ \oplus \Lambda^2_- \to \Lambda^2_+ \oplus \Lambda^2_-$$

where Tr(A)=Tr(D)= S/4 where S is the scalar curvature, B corresponds to the traceless Ricci tensor and the traceless part of A and D are the SD and ASD Weyl tensors, $W^\pm$ respectively. Identifying $\mathcal{A}$ with Λ^2_-, the natural definition of QK manifold in dimension 4 becomes that M is an oriented Riemannian manifold which is Einstein ($B = 0$) and self-dual ($W^- = 0$.)

We now move on to describe the twistor space construction for HK and QK manifolds in the same spirit as we did for the 4 dimensional case.

For a general HK manifold, (M, g, I, J, K) of dimension $4n$ we simply define its twistor space, Z to be the trivial bundle, $M \times S^2$. This bundle is trivial due to the fact that we have a global basis of the bundle $\mathcal{A}$ and hence restricting to the unit sphere in each fibre gives a trivial $\mathbb{P}^1$ bundle. As before we can define a natural almost complex structure on Z which in this situation can be described more explicitly. Using the local coordinates on the open set, $\{[1 : z] \mid z \in \mathbb{C}\}$ we can parametrise the fibres as;

$$(a, b, c) = (\frac{1 - |z|^2}{1 + |z|^2}, \frac{2\mathrm{Re}(z)}{1 + |z|^2}, \frac{2\mathrm{Im}(z)}{1 + |z|^2})$$

where we are using I, J and K as the standard Euclidean coordinates. So the natural almost complex structure at $z = (m, (a, b, c)) \in Z$ is given by

$$Q = (aI + bJ + cK) \oplus i.$$

where multiplication by i is just the complex structure on $\mathbb{P}^1$. As in the 4 dimensional case Q is integrable. This is proved directly using the Newlander-Nirenberg theorem. We need to check that the algebraic ideal generated by the holomorphic 1-forms is in fact a differential ideal i.e.

$$d\theta = \theta_i \wedge \alpha_i$$

where $Q\theta = i\theta$, θ_i are a basis of holomorphic 1 forms and α_i are arbitrary 1-forms. In the given local coordinates the $(1, 0)$-forms for Q are spanned by $\beta_i + zK\beta_i$ for $(1, 0)$ forms β_i on

M w.r.t I. It is then an easy matter of computation to check that the ideal is closed under exterior derivative. We now list some features of Z which are very similar to those described in the previous section that will allow a reverse generalised Penrose construction.

The normal bundle of each fibre, $\pi^{-1}(m)$ is obviously just $T_m M$ which as a holomorphic vector bundle is $\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$ ($2n$ times) (we show an explicit computation of this for a specific case in the next section).

Right multiplication by j on $\mathbb{C}^2 \cong \mathbb{H}$ induces a real structure on $\mathbb{P}^1$. This induces an anti-holomorphic involution on the fibres which can equivalently be described as the anti-podal map on S^2.

The $(1,0)$ forms on Z can be used to define a complex $(2,0)$ form which restricts to a holomorphic symplectic form on the fibres for the projection, $p : Z \to \mathbb{P}^1$. As a holomorphic form, it is a section of the bundle $\Lambda^2(\ker dp)^* \otimes \mathcal{O}(2)$. We can state a generalised Penrose theorem which recovers M from Z.

Theorem 5.1. *[6] Suppose that Z is a complex manifold of real dimension $4n+2$ so that the projection map, $p : Z \to \mathbb{P}^1$ is a holomorphic fibre bundle that has a family of holomorphic sections with normal bundle $\mathcal{O}(1) \otimes \mathbb{C}^{2n}$ and there is a holomorphic 2-forms and involution map as described above. Then the parameter space of real sections is a 4n-dimensional HK manifold, M with Z as its twistor space.*

The QK case is quite different due to fact that the bundle is not trivial. The twistor space is defined as the sphere bundle in $\mathcal{A}$ consisting of almost complex structures compatible with the metric. By the definition of QK manifolds, the Levi Civita connection preserves the bundle $\mathcal{A}$ in the sense that $\nabla I, \nabla J$ and ∇K are combinations of I, J and K. Hence we once again have a splitting of the tangent space into a horizontal and vertical component. The description for the construction of the almost complex structure is similar as that described in the 4-dimensional case and turns out to be integrable as well. The only difference being that the base manifold need not be SD.

6 An explicit description of the twistor space of $\mathbb{P}^2$

Definition 6.1. *Let $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$ be a sequence of subspaces of a vector space, V strictly contained in each other, with $d_i = \dim(V_i)$. This is called a flag in V. For fixed choice of $(d_1, ..., d_k)$, the set of all flags defines a flag manifold.*

The flag manifold, $\mathbb{F}^{(1,2,3)}$ is the space of flags of type $(1,2,3)$ in $\mathbb{C}^3$. As a homogeneous space it is given by,

$$\frac{U(3)}{U(1) \times U(1) \times U(1)}$$

which we shall come back to in the next section.

Noting that once a choice of a vector in $\mathbb{C}^3$ has been made there is only a choice of a second vector in its orthogonal complement $\mathbb{C}^2$ to define the flag, we get the equivalent description of $\mathbb{F}^{(1,2,3)}$ as a projective variety,

$$\mathbb{F}^{(1,2,3)} = \{(v, w) \in \mathbb{P}^2 \times \mathbb{P}^2 \mid v \cdot w = 0\}.$$

The twistor map, $\pi : \mathbb{F}^{(1,2,3)} \to \mathbb{P}^2$ is then given by

$$\pi([v], [w]) = [\bar{v}_2 w_3 - \bar{v}_3 w_2 : \bar{v}_3 w_1 - \bar{v}_1 w_3 : \bar{v}_1 w_2 - \bar{v}_2 w_1]$$

In order to check that this is indeed a twistor map, we need to verify that the fibres are $\mathbb{P}^1$, the normal bundle of the fibres are $\mathcal{O}(1) \oplus \mathcal{O}(1)$ and that there is an anti-holomorphic involution preserving the fibres. Since we just saw that it is a homogeneous space, it is sufficient to check the properties for a fixed fibre.

Considering the point $[1:0:0] \in \mathbb{P}^2$, an easy computation shows that

$$\pi^{-1}[1:0:0] = \{([0:1:\bar{z}], [0:-z:1]) \mid z \in \mathbb{C}\} \cup \{[0:0:1], [0:1:0]\}$$

which shows that indeed the fibre is a sphere. Observe that the anti-holomorphic involution map given by the map, $([x:y:z], [u:v:w]) \mapsto ([u:v:w], [x:y:z])$ indeed defines an automorphism of the fibres.

We find explicitly the normal bundle of this $\mathbb{P}^1$.

Consider the (inverse) chart maps, $\phi_1, \phi_2 : \mathbb{C}^3 \to \mathbb{F}^{(1,2,3)}$ given by,

$$\phi_1(x, y, z) = ([\bar{x} : 1 : \bar{y}], [z : -y - xz : 1])$$
$$\phi_2(x, y, z) = ([\bar{x} : \bar{y} : 1], [z : 1 : -y - xz])$$

in which the fibre $\mathbb{P}^1$ is covered by

$$\phi_1(0, y, 0) = ([0 : 1 : \bar{y}], [0 : -y : 1])$$
$$\phi_2(0, y, 0) = ([0 : \bar{y} : 1], [0 : 1 : -y]).$$

We compute the transition maps as,

$$(\phi_2^{-1} \circ \phi_1)(x, y, z) = (\frac{x}{y}, \frac{1}{y}, \frac{z}{-y - xz})$$

and restricting to $\mathbb{P}^1$ i.e. $x = z = 0$ we get

$$\begin{aligned}(\phi_2^{-1} \circ \phi_1)_* \partial_x &= \frac{1}{y} \partial_x \\ (\phi_2^{-1} \circ \phi_1)_* \partial_z &= -\frac{1}{y} \partial_z\end{aligned}$$

which corresponds indeed to the transition maps of $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This shows that indeed the flag manifold $\mathbb{F}^{(1,2,3)}$ is the twistor space of $\mathbb{P}^2$. Note that $\mathbb{P}^2$ is an Einstein manifold with the Fubini-Study metric and is SD hence a QK 4-manifold.

7 Symmetric spaces

Definition 7.1. *A Riemannian symmetric space, (M, g) has at each point, $p \in M$ an isometry, $f_p : M \to M$ such that p is a fixed point and $f(\exp_p(X)) = \exp_p(-X)$ for every $X \in T_pM$.*

If we let G be the Lie group of isometries of M acting transitively and H the stabiliser of a point, p then we have that M and G/H are diffeomorphic. Furthermore, the map f_p defines an involution, $\sigma : G \to G$ by $\sigma(g) = f_p \circ g \circ f_p$.

A general triple (G, H, σ) where H is a closed subgroup of the fixed point set of an involution σ containing the identity component is called a symmetric space. The differential of the involution gives an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

with eigenvalues +1 and -1 respectively. We have that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$ and $ad(H)\mathfrak{m} \subset \mathfrak{m}$. The last condition means that $\mathfrak{g}$ is a reductive Lie algebra.

In the Riemannian case, we have that H has a compact isotropy representation on T_pM since it is a subgroup of $O(n)$.

One can recover the Riemmanian symmetric manifold, (M, g) from the data (G, H, σ) when H has a compact isotropy representation. We define M as the coset space G/H. The involution σ defines the symmetries on the coset space. Since $Ad(H)$ is compact we can define an H-invariant inner product on $T_o(G/H)$ with respect to which $\mathfrak{h}$ and $\mathfrak{m}$ are orthogonal. This is done simply by choosing an arbitrary inner product on $\mathfrak{g}$ and integrating over H to obtain an H-invariant inner product. Restricting this inner product to $\mathfrak{m}$ which can naturally be identified with $T_o(G/H)$ we extend it to M by acting on the left by G. Any such invariant metric on G/H is in fact isometric to (M, g).

If the symmetric space (M, g) also has an almost complex structure J compatible with the metric such that $f_{p*} \circ J = J \circ f_{p*}$ then M is said to be a Hermitian symmetric space. The fact that ∇J is a degree 3 tensor and invariant by the symmetries implies that $\nabla J = 0$. Hence any Hermitian symmetric space is necessarily Kähler. Recovering the Hermitian symmetric space from (G, H, σ) is again possible if $\mathfrak{m}$ admits an almost complex structure J which commutes with $Ad(H)$ and then extending it to a left invariant almost complex structure on G/H. In this case, J is also integrable.

We should mention here that the notion of symmetric spaces hold whenever we have a manifold endowed with a connection ∇ and in such a situation we only need the symmetry maps to preserve ∇. The Riemannian case is just the special case when ∇ is the Levi Civita connection and this automatically implies that the symmetries are isometries since ∇ is determined by g itself and the fact that it is torsion free.

We are interested in the case when $G = SU(3)$ and H is either $S(U(1) \times U(2))$ or $S(U(1) \times U(1) \times U(1))$. The Hermitian symmetric spaces are then:

$$\mathbb{P}^2 \cong \frac{SU(3)}{S(U(1)\times U(2))}, \qquad \mathbb{F}^{(1,2,3)} \cong \frac{SU(3)}{S(U(1)\times U(1)\times U(1))}$$

We have shown in the previous section that $\mathbb{F}^{(1,2,3)} \to \mathbb{P}^2$ is a $\mathbb{P}^1$ fibration. Recall that the Fubini-Study metric on $\mathbb{P}^2$ can be defined as follows. Consider the real valued function, $f : \mathbb{C}^3 - \{0\} \to \mathbb{R}$ given by $f(z_1, z_2, z_3) = \log(\Sigma|z_i|^2)$. Then the $(1,1)$-form $\omega = \partial\bar{\partial} f$ is horizontal for the projection to $\mathbb{C}^3/\mathbb{C}^\times$ and is invariant under the action of $\mathbb{C}^\times$. So it descends to a 2-form on $\mathbb{P}^2$ which pairs with the natural complex structure to give the Fubini-Study metric, g, which can written in local coordinates on $\{[1 : x+iy : u+iv]|x, y, u, v \in \mathbb{R}\}$ as

$$g = \begin{pmatrix} 1+u^2+v^2 & -xu-yv & 0 & -xv+yu \\ -xu-yv & 1+x^2+y^2 & xv-yu & 0 \\ 0 & xv-yu & 1+u^2+v^2 & -xu-yv \\ -xv+yu & 0 & -xu-yv & 1+x^2+y^2 \end{pmatrix}$$

Note that since f is $SU(3)$ invariant hence so is g. It should therefore not come as a surprise that this metric will coincide exactly with the canonical metric on $SU(3)/S(U(1)\times U(2))$. In other words, the canonical metric on the symmetric space described above is exactly the Fubini-Study metric. The same argument holds for the Flag manifold. We have therefore reduced our problem to one about symmetric spaces.

We choose the following basis for $\mathfrak{su}(3)$

$$v_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} v_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} h_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} h_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$h_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} h_4 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} q_1 = \begin{pmatrix} i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -i \end{pmatrix} q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$$

Hence the Maurer Cartan form on $SU(3)$ takes the form,

$$\omega = \begin{pmatrix} ie^7 & e^1+ie^3 & e^2+ie^4 \\ -e^1+ie^3 & ie^8 & e^5+ie^6 \\ -e^2+ie^4 & -e^5+ie^6 & -i(e^7+e^8) \end{pmatrix}$$

where e^j are the left invariant 1-forms dual to our choice of basis for $\mathfrak{su}(3)$ and we compute

$$d\omega = -\begin{pmatrix} 2i(13+24) & \begin{matrix}(37+83-25+64) \\ +i(71+18+26-45)\end{matrix} & \begin{matrix}(15+47+63+47+48) \\ +i(72+16+35-27-28)\end{matrix} \\ \cdot & 2i(31+56) & \begin{matrix}(21+43+68+67+68) \\ +i(41+32+85+75+85)\end{matrix} \\ \cdot & \cdot & 2i(42+65) \end{pmatrix}$$

where we denoted $e^i \wedge e^j$ by ij simply for convenience. We can view $SU(3)$ as a distinguished frame bundle on $\mathbb{F}^{(1,2,3)}$ with structure group $U(1) \times U(1) \subset U(3) \subset SO(6)$. This means in particular that it determines a metric and a compatible almost complex structure. The associated connection form is simply ω without the off-diagonal terms and the the matrix without the diagonal elements is the solder form. In other words the Cartan connection provides an absolute parallelism. Under the bundle isomorphism $T\mathbb{F}^{(1,2,3)} \cong SU(3) \times_{ad} (\mathfrak{su}(3)/\mathfrak{u}(1)^2)$ given by the solder form, we get the metric,

$$g_{_{\mathbb{F}(1,2,3)}} = e^1e^1 + e^2e^2 + e^3e^3 + e^4e^4 + 2e^5e^5 + 2e^6e^6$$

and the Kähler form,

$$\Omega_{_{\mathbb{F}(1,2,3)}} = e^{13} - e^{24} - 2e^{56}$$

where we still denote by e^i the pullback of the horizontal forms to $\mathbb{F}^{(1,2,3)}$. The fact that $d\Omega = 0$ is checked directly from the above given relations. The almost complex structure associated to this structure is the integrable one that we described earlier which maps e^5 to e^6. The non-integrable complex structure is thus given by reversing the map on the vertical vectors. According to a theorem by Bérard Bergery (see [2]) there exists exactly two Einstein metrics on the flag manifold which can arise by a canonical variation, which we find as follows. By rescaling the metric on the fibres, we hence get the nearly Kähler form on the twistor space of $\mathbb{P}^2$ as

$$\hat{\Omega}_{_{\mathbb{F}(1,2,3)}} = e^{13} - e^{24} + e^{56},$$

in which case we have

$$d\hat{\Omega}_{_{\mathbb{F}(1,2,3)}} = 3((e^{12} + e^{34})(-e^6) + (e^{14} + e^{23})e^5)$$

This 2-form naturally corresponds to the non-integrable complex structure we described earlier. Our example fits into a class of manifolds called nearly Kähler manifolds introduced by Alfred Gray.

Definition 7.2. *An almost Hermitian manifold,* (M, g, J, ω) *is said to be nearly Kähler if* ∇J *is skew-symmetric i.e.* $\nabla_X JX = 0$ *for all* $X \in \Gamma(TM)$.

These manifolds are said to have weak holonomy group [5] $U(n)$, which geometrically means that given any loop, γ in M with $X = \gamma'(o)$ the parallel transport of the holomorphic section containing X along γ takes values in $U(n)$. The equivalence of these definitions is seen from the formula

$$(\nabla_X J)X = \nabla_X(JX) - J(\nabla_X X)$$

together with the fact that $U(n)$ and J commute.Compact nearly Kähler 6-manifolds, sometimes referred to as Gray manifolds, can also be defined as 6-manifolds with structure group, $SU(3)$ defined by the addition of a holomorphic volume form, $\psi^+ + i\psi^-$, which satisfy the relations

$$d\omega = 3\psi^+,$$
$$d\psi^- = -2\omega \wedge \omega.$$

In our example the holomorphic volume form is defined by

$$\psi^+ + i\psi^- = (1 + i3) \wedge (4 + i2) \wedge (5 + i6)$$

and a direct computation yields

$$\begin{aligned} d\psi^- &= 4(6425 + 1432 + 3156) \\ &= -2\ \hat{\Omega}_{\mathbb{F}^{(1,2,3)}} \wedge \hat{\Omega}_{\mathbb{F}^{(1,2,3)}} \end{aligned}$$

$$d\hat{\Omega}_{\mathbb{F}^{(1,2,3)}} = 3\psi^+$$

which shows that $\mathbb{F}^{(1,2,3)}$ with the anti-canonical complex structure is indeed nearly Kähler. We end the second part of this joint project here.

References

[1] Michael Atiyah, Nigel Hitchin and Isadore Singer, *Self duality in 4-dimensions,* Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences Vol. 362, No. 1711 (1978), pp. 425-461

[2] Arthur L. Besse, *Einstein Manifolds,* Classics in Mathematics ISSN 1431-0821, 1987, Springer-Verlag.

[3] Claude LeBrun, *Geometry of Twistor Spaces,* Simons Workshop Lecture, 7/30/04, Lecture Notes by Jill McGowan.

[4] J. Eells and S. Salamon, *Twistor construction of harmonic maps of surfaces into 4-manifolds,* Annali della Scuola Normale Superiore di Pisa, 1985, p.589-640

[5] A. Gray, *Nearly Kähler Manifolds,* Journal of Differential Geometry, 1970, p.283-309

[6] N. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkähler Metrics and Supersymmetry,* Communications in Mathematical Physics, 1987, 108 p.533-589

[7] Katsumi Nomizu and Shoshichi Kobayashi, *Foundations of differential geometry volume 1 and 2,* John Wiley & Sons, 1969

[8] R. Penrose, *Nonlinear gravitons and curved twistor theory,* Gen. Relativity. Grav., 1976, p.31-52

[9] Michael Spivak, *A comprehensive introduction to differential geometry,* Publish or Perish Inc., 1970

[10] A. Haydys, *HyperKähler and quaternionic Kähler manifolds with S^1 -symmetries,* Journal of Geometry and Physics, 2008

[11] N. Hitchin, *On the Hyperkähler/Quaternion Kähler Correspondence,* Communications in Mathematical Physics, 2013

[12] M. Gross and P. Wilson, *Large Complex Structure Limits of K3 Surfaces,* Mathematics - Differential Geometry, Mathematics - Algebraic Geometry, 14J32, 2000

[13] A. Swann, *Hyperkähler and Quaternionic Kähler Geometry,* Phd thesis, Oxford University, 1990

[14] L. Foscolo, *On moduli spaces of periodic monopoles and gravitational instantons,* PhD thesis, Imperial College London, 2013

[15] M. Kalafat and J. Sawon, *Hyperkähler manifolds with circle actions and the Gibbons-Hawking Ansatz,* Mathematics - Differential Geometry, Mathematical Physics, 53C26, 83C20, 2009